2021

MATHEMATICS — HONOURS

Second Paper

(Module – III)

Full Marks : 50

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

$\mathbb{R}$, $\mathbb{Q}$, $\mathbb{N}$ denote the sets of real numbers, rational numbers and natural numbers respectively.

Group – A

(Marks : 40)

Answer ***any four*** questions.

1. (a) State and prove Archimedean property of real numbers.

(b) Find Sup A and inf A, if exist, where $A=\left\{x\in\mathbb{R} : x^2-3x-10<0\right\}$.

(c) Define neighbourhood of a point in $\mathbb{R}$. Check whether the set $\left\{\frac{1}{n} \middle| n\in\mathbb{N}\right\}\cup\{0\}$ is a neighbourhood of 0 or not. (1+3)+3+(1+2)

2. (a) Show that [0, 1] is not enumerable.

(b) Prove or disprove : A countable set cannot have uncountable number of limit points.

(c) Show that the set $\{x\in\mathbb{R} : \sin x=0\}$ is countable. 4+3+3

3. (a) Let the sequence $\{x_n\}$ be such that $x_n\neq x_m$ for $n\neq m$. Prove that every subsequential limit of $\{x_n\}$ is a limit point of $\{x_n : n\in\mathbb{N}\}$.

(b) A sequence $\{x_n\}$ is as follows : $x_2\le x_4\le x_6\le .....\le x_5\le x_3\le x_1$ and $\{y_n\}$ is defined by $y_n=x_{2n-1}-x_{2n}, n\in\mathbb{N}$ such that $\{y_n\}$ converges to zero. Show that $\{x_n\}$ is convergent.

(c) Show that arbitrary union of open subsets of $\mathbb{R}$ is an open set. 3+4+3

4. (a) Define Cauchy sequence. Prove that every Cauchy sequence is bounded. Is the converse true? Justify.

(b) Prove or disprove : The sequence $\{x_n\}$ where $x_n = n/2 - [n/2]$, is convergent, $[x]$ denotes the largest integer not exceeding x.

(c) Prove that if $\lim_{n\to\infty} x_n = l$, then for any $m > 0$, $\lim_{n\to\infty} (m x_n) = m.l.$ (1+3+2)+2+2

5. (a) Let $f : \mathbb{Q} - \{0\} \to \mathbb{R}$ be defined by $f(x) = x^2 \sin\frac{1}{x}, \forall x \in \mathbb{Q} - \{0\}$. Does $\lim_{x\to 0} f(x)$ exist? Justify your answer.

(b) If $\{u_n\}_n$ and $\{v_n\}_n$ be bounded sequences of real numbers.

Prove that $\overline{Lim}\, u_n + \overline{Lim}\, v_n \geqslant \overline{Lim}\, (u_n + v_n)$.

(c) Prove that every sequence of real numbers has a monotone subsequence.

(d) Prove or disprove : If $\{x_n\}$ is a bounded sequence and $\{y_n\}$ is a convergent sequence then $\{x_n y_n\}$ is a convergent sequence. 2+2+4+2

6. (a) Let $f : D \to \mathbb{R}$ ($D \subseteq \mathbb{R}$) be a function such that $\lim_{n\to\infty} f(x_n) = f(a)$, for any sequence $\{x_n\}$ in D converging to $a \in D$. Show that f is continuous at a.

(b) Correct or Justify : There exists a monotonic function defined on [0, 1] such that the function is discontinuous at every irrational point in [0, 1].

(c) If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, prove that f is bounded on $[a, b]$.

(d) Let f, g be two real valued continuous functions of real variable and $f(x) = g(x), \forall x \in \mathbb{Q}$. Show that $f(x) = g(x), \forall x \in \mathbb{R}$. 3+2+3+2

7. (a) Define Lipschitz function. Show that every Lipschitz function is uniformly continuous.

(b) Let $f : [0, 1] \to [0, 1]$ be a continuous function. Using Intermediate Value Property of f, show that there exists a point $c \in [0, 1]$ such that $f(c) = c^2$.

(c) Prove or disprove : A sequence is everywhere continuous.

(d) Prove or disprove : the function $f(x) = x \sin 1/x,\ x \neq 0$, is uniformly continuous on $(0, 1/\pi)$.

(1+2)+3+2+2

Group – B

(Marks : 10)

8. Answer ***any two*** questions :

(a) If $I_{m,n} = \int \sin^m x \cos^n x\, dx$, where m, n are positive integers, then prove

$$I_{m,n} = \frac{\sin^{m-1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n} I_{m,\, n-2}.$$

Hence find a reduction formula for $I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x\, dx.$ 4+1

(b) Find the value of $\int_0^1 \frac{dx}{(3+4\cos x)^2}.$ 5

(c) Evaluate : $\int \frac{2x dx}{\left(1-x^2\right)\sqrt{x^4-1}}$ 5

(d) Reduce $\lim_{n\to\infty}\left[\frac{1}{n} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + \ldots.. + \frac{1}{8n}\right]$ to a definite integral and hence find its value.

2+3