M.Sc. (Physics) 4th Semester 2020
PHY 521 (Quantum Field Theory)

Full marks: 50 Time: 2.5 hrs

(*2 hours for answering and 30 minutes for downloading, scanning, and mailing back*)

(i) Please write your Examination Roll Number and Registration Number (from an earlier admit card) at the top of your answer script. Do not write your name or class roll number anywhere.
(ii) Scan the complete answer script into a single pdf file and mail it to the e-mail from where you got the paper. The answer script file must be named as XXXXXXPHY521.pdf, where XXXXXX is the university roll number. For example, the script coming from C91/PHS/181099 must be named **181099PHY521.pdf**.

*Answer any **five** questions.*

1. (a) Consider a real scalar field with the following Lagrangian density:

$$\mathcal{L} = \frac{1}{2}(\partial^\mu\phi)(\partial_\mu\phi) - \frac{1}{2}m^2\phi^2 - \frac{g}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4 \,.$$

(i) Draw a Feynman diagram for $\phi\phi \to \phi\phi\phi$ at the tree-level (any one of the possible diagrams will do). Find the amplitude, including the symmetry factor.
(ii) Suppose ϕ is in a 3-dimensional (*i.e.* 2 space and 1 time) space-time, instead of the usual 4. In that case, what should be the mass dimension of g and λ?

(b) Assuming the Wick's theorem to be true for n fields, show that it must be true for $n+1$ fields. Take a real scalar field to prove this.

(c) The electron-photon interaction term in QED is written as $-e\bar{\psi}\gamma^\mu\psi A_\mu$. Show, from Wick's theorem, that any amplitude with an electron, a positron, and a photon as external legs must be proportional to an odd power of the electric charge e.

(3+2) + 3 + 2

2. (a) Starting from the Lagrangian density of the free electromagnetic field $\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$, show that the photon propagator cannot be defined unless the gauge is fixed.

(b) Consider the $e^+e^- \to e^+e^-$ scattering in QED. Suppose you write the photon propagator as

$$-\frac{i}{q^2}\left(\eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) ,$$

q being the four-momentum of the photon. Show that the contribution of the second term ($\propto q_\mu q_\nu$) to the amplitude vanishes. Keep the electron mass for your calculation.

5 + 5

3. (a) The bilinear covariants are written as $\bar{\psi}\Gamma\psi$, where Γ can be any one of the 16 matrices:

$$\Gamma = \{\mathbf{1}, \gamma^\mu, \sigma^{\mu\nu}, \gamma^\mu\gamma_5, \gamma_5\} \,.$$

Show that

$$\sum_{i=1}^{16} a_i\Gamma_i = 0$$

necessarily means all a_i to be equal to zero. You can use the trace theorems for the γ-matrices.

(b) The matrix $\sigma^{\mu\nu}$ is defined as $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. Show that

$$\frac{i}{2m}\,\bar{u}_f\,\sigma^{\mu\nu}(p_f - p_i)_\nu\,u_i = \bar{u}_f\,\gamma^\mu\,u_i - \frac{1}{2m}\,\bar{u}_f\,(p_i + p_f)^\mu\,u_i\,,$$

where u_i and u_f are on-shell momentum space spinors for some fermion of mass m, with four-momenta p_i and p_f respectively.

5 + 5

4. (a) Show that the free propagator for the real scalar field ϕ is the Green's function for the Klein-Gordon equation, *i.e.*,

$$\left(\partial_\mu \partial^\mu + m^2\right)\,\Delta(x - x') = -i\delta^4(x - x')$$

with $\Delta(x - x') = \langle 0|\mathcal{T}(\phi(x)\phi(x')|0\rangle$. The symbols carry their usual meanings.

(b) Draw the Feynman diagrams (at the lowest order) for the scattering $e^-\gamma \to e^-\gamma$. Show that the total amplitude is invariant under $U(1)_{\rm em}$ gauge transformation. Do not neglect the electron mass.

5 + (1+4)

5. (a) Find the expression for the field strength tensor $F^a_{\mu\nu}$ from $[D_\mu, D_\nu] = igT^aF^a_{\mu\nu}$, where D_μ is the covariant derivative of SU(N), g is the SU(N) coupling constant and T^a's are the SU(N) generators.

(b) Consider the Lagrangian $\mathcal{L} = \bar{\psi}\,(i\gamma_\mu D^\mu - m)\,\psi$ where $D_\mu = \partial_\mu + igT_aG^a_\mu$ and g is the gauge coupling. Make a local transformation $\psi \to \psi' = U(x)\psi$, $G^a_\mu \to G'^a_\mu$. Show that the condition for invariance of $\mathcal{L}$ is $T_aG'^a_\mu = \frac{i}{g}\,(\partial_\mu U)\,U^{-1} + UT_aG^a_\mu U^{-1}$. Use an infinitesimal transformation $\psi' = \psi - ig\beta_a(x)T^a\psi$ to show that $G'^a_\mu = G^a_\mu + \partial_\mu\beta^a + gf^{abc}\beta_bG_{\mu c}$ where $[T_a, T_b] = if_{abc}T_c$.

4 + (3+3)

6. (a) An infinitesimal proper Lorentz transformation (on a 4-vector) can be written as $\Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu$. Show that $\omega^{\mu\nu}$ is an antisymmetric tensor. How many independent components does it have? Show that the above set of transformations form a group.

(b) Write down the commutators $[K_i, K_j]$ and $[J_i, K_j]$, where J_i and K_i are the rotation and the boost generators respectively. Given $J_{\pm i} = \frac{1}{2}(J_i \pm iK_i)$, what are K_is for the $(\frac{1}{2}, 0)$ representation and the $(0, \frac{1}{2})$ representations? Which operation interchanges the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations?

5 + 5

7. (a) Consider the Lagrangian for a complex scalar field :

$$\mathcal{L} = (\partial_\mu\phi)^\dagger(\partial^\mu\phi) - m^2\phi^\dagger\phi - \frac{\lambda}{4!}(\phi^\dagger\phi)^2\,,$$

with $m^2 < 0$ and $\lambda > 0$. Let $|\langle 0\,|\,\phi\,|\,0\rangle| = v$. Check that redefining $\phi = v + (\xi + i\eta)/\sqrt{2}$ would lead to a real massless scalar field and a massive neutral scalar field with correct sign of the mass term.

(b) Consider now the $U(1)$ gauged version of the above Lagrangian, called scalar electrodynamics. Show how a gauge field picks up mass due to spontaneous symmetry breaking.

4 + 6