## 2019

## PHYSICS

> Module: PHY-423
(Statistical Mechanics)
Full Marks : 50
The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

Answer any five questions.
(a) The low energy excitations of a quantum ferromagnet are waves called magnons (which are bosons). Magnons have a dispersion relation $\omega(k)=A|k|^{2}$ where $A$ is a constant. Magnons have only one polarisation state and their number is not conserved. Show that the internal energy $U \sim T^{52}$. All the symbols have their usual meaning.
(b) Starting from the distribution functions of BE and FD statistics show that the number of particles at zero energy can be considerably large for Bosons whereas it is not so for Fermions. You may assume, here, the chemical potential to be zero in both cases.
(c) In an ensemble of spin $1 / 2$ particles, $1 / 4$ of the spins are in the $s^{2}=-1 / 2$ state while the rest are in the $s^{x}=1 / 2$ state. Evaluate the density matrix for the system.
$3+2+5$
(a) Write down the partition function for a system of $N$ non-interacting particles contained in volume $V$ at temperature $T$ in terms of configuration integral, explaining all the symbols. Hence write all the terms of configuration integral explicitly for $N=3$. Draw the 3 -particle graphs corresponding to each term, specifying their $\left\{m_{l}\right\}$ values.
(b) The three lowest energy levels of a certain molecule are $E_{1}=0, E_{2}=\epsilon, E_{3}=10 \epsilon$. Show that at sufficiently low temperatures (how low?) only levels $E_{1}$ and $E_{2}$ are populated. Find the average energy $E$ of the molecule at temperature $T$. Find the contributions of these levels to the specific heat per mole $C_{v}$.
(a) What is the maximum value of the chemical potential for bosons? Justify your answer.
(b) The quantum states available to a given physical system are (i) a group of $g_{1}$ equally likely states, with a common energy $E_{1}$ and (ii) a group of $g_{2}$ equally likely states, with a common energy $E_{2}-E_{1}$. Show that this entropy of the system is given by $S=-k\left[p_{1} \ln \left(p_{1} / g_{1}\right)+p_{2} \ln \left(p_{2} g_{2}\right)\right]$. where $p_{1}$ and $p_{2}$ are, respectively, the probabilities of the system being in a state belonging to group 1 or to group $2 ; p_{1}+p_{2}=1$.
(c) Consider a Bose gas in 2 dimension with surface area $A$. Derive an expression for the number of particles in the excited state $\left(N_{e}\right)$ and in the ground state $\left(N_{0}\right)$ as a function of $y\left(=e^{\mu / k T}\right), T$ and $A$. Show that the system does not exhibit BE condensation unless $T \rightarrow 0$. You may use

$$
\int_{0}^{\infty} \frac{d x}{y^{-1} e^{x}-1}=-\ln (1-y)
$$

4. (a) Consider a one dimensional spin-1/2 Ising ring in the absence of an external magnetic field. The Hamiltonian is

$$
\mathcal{H}=-J \sum_{i=1}^{N} S_{i} S_{i+1}
$$

Here $S_{i}= \pm 1$. Construct the transfer matrix and solve for its eigenvalues.
(b) For a Ising model in 2-dimension with nearest-neighbour interaction energy $J_{1}$ and second-nearest neighbour interaction energy $J_{2}$, the renormalization group recursion relations (neglecting all the higher order interactions terms) are found to be:

$$
\begin{aligned}
& K^{\prime 2}=2 K^{2}+L \\
& L^{\prime}=K^{2} .
\end{aligned}
$$

Here $K=J_{1} / k_{B} T$ and $L=J_{2} / k_{B} T$.
Solve the equations simultaneously to show that it corresponds to 3 fixed points : (i) $\left(K^{*}, L^{*}\right)=(0,0)$ (ii) $\left(K^{*}, L^{*}\right)=(\infty, \infty)$ and (iii) $\left(K^{*}, L^{*}\right)=(1 / 3,1 / 9)$. What are the spin configurations corresponding to the first two fixed points? Argue that the system can undergo a phase transition at a non-zero critical temperature.
5. (a) Consider a square lattice, consisting of $A$ atoms and $B$ atoms. In the disordered phase, both the atoms can occupy a lattice site with equal probabilities. In the ordered phase, they are alternately arranged in a chess board pattern i.e., one sublattice is completely occupied by $A$ atoms and the other, by $B$ atoms.
Identify the broken symmetry corresponding to this phase transition and construct an appropriate order parameter.
(b) Let the critical exponents $\beta$ and $\delta$ be defined as:
(i) $H=0, t \rightarrow 0, M \sim|t|^{\beta}$
(ii) $t=0, H \rightarrow 0, M \sim|H|^{1 / \delta}$

Here $t=\left(T-T_{c}\right) / T_{c}, M$ is the order parameter, and $H$ is the conjugate field.
Now consider a system with a modified expression for the Landau free energy :

$$
f(M)=\frac{1}{2} r M^{2}+s M^{4}+u M^{6}-H M
$$

If $u>0$ and $r=a\left(T-T_{c}\right)$, evaluate $\beta$ and $\delta$ for the special case when $s=0, \quad(2+2)+(3+1$
6. (a) Find the critical exponents corresponding to the following functions :
(i) $f(t)=1.5 t^{1 / 2}+0.5 t^{-1 / 3}$.
(ii) $f(t)=a t^{-2 / 3}(t+b)$

Here $t=\left(T-T_{c}\right) / T_{c}$, which measures the deviation of the temperature $T$ from the critical value $T_{c}$.
(b) Show that for a system in canonical ensemble :

$$
\begin{equation*}
\left\langle(\Delta E)^{3}\right\rangle=k^{2}\left[T^{4}\left(\frac{\partial C_{V}}{\partial T}\right)_{V}+2 T^{3} C_{V}\right] \tag{2+2}
\end{equation*}
$$

where the symbols have their usual meaning.
7. (a) From the solution of the Langevin equation:

$$
v(t)=v_{0} e^{-\gamma t}+\frac{1}{m} \int_{0}^{t} \eta\left(t_{1}\right) e^{-\gamma\left(t-t_{1}\right)} d t_{1}
$$

(Here $\eta(t)$ is a random force and $\gamma$ is the coefficient of friction) compute $\langle v(t)\rangle$ and $\left\langle v^{2}(t)\right\rangle$. Show that if there is no dissipation in the system, that will lead to an unphysical consequence.
(b) Consider the spontaneous fluctuations in an equilibrium system, denoted by the parameter $A$. Let $\delta A(t)=A(t)-\langle A\rangle$ be the instantaneous deviation of $A(t)$ from its time-independent equilibrium average. $C(t)=\langle\delta A(0) \delta A(t)\rangle$ is the corresponding time-correlation function. Argue that
(i) $C(t)=C(-t)$
(ii) $C(t) \rightarrow 0$ when $t \rightarrow \infty$

