M.Sc. (Physics) 4th Semester 2020
PHY 523 (Nonlinear Dynamics)

Full marks: 50 Time: 2.5 hrs

(*2 hours for answering and 30 minutes for downloading, scanning, and mailing back*)

Instructions:

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Answer any *five* questions.

1. (a) Briefly explain the Existence and Uniqueness Theorem as applicable to one dimensional systems. As an illustration of this theorem, find all solutions of the initial value problem,

$$\frac{dx}{dt} = \frac{5}{3}\,x^{2/5}\,, \qquad \text{with} \qquad x(t=0) = 0.$$

Schematically plot the solution(s). What would the solution(s) be for $x(t=0) = x_0 \neq 0$.

(b) $\dot{x} = f(x)$ represents a one dimensional autonomous system where $f(x)$ is assumed to be a smooth function. As an example, for $f(x) = \sin(x)$, there exist fixed points at $x^* = \pm n\pi$, with $n = 0, 1, 2, \ldots$.
For each of the following cases, find an $f(x)$ with the stated properties. If there are no such $f(x)$, explain why.

i. Every real number is a fixed point.
ii. There are precisely three fixed points and all of them are stable.
iii. There are no fixed points.

(1+3+1)+(2+2+1)

2. (a) Consider the equation $\dot{x} = r\,x + x^3$, where $r > 0$ is fixed. Show that the solution $x(t) \to \pm\infty$ in finite time, starting from any initial condition $x(0) \neq 0$. Clearly mention any approximation(s) made.

(b) Consider the system $\dot{x} = r\,x + x^3 - x^5$.

i. Obtain algebraic expressions for all the fixed points as r takes different values from negative to positive, through zero.
ii. For different r values, sketch the vector fields. Clearly indicate all the fixed points and their stability.
iii. Find the value of r at which there is a saddle-node bifurcation.

3+(3+3+1)

3. (a) Consider the two dimensional system $\dot{x} = a\,x$, $\dot{y} = -y$, where $a < -1$. Argue, using a schematic plot or otherwise, that all trajectories become parallel to the y-direction as $t \to +\infty$, and parallel to the x-direction as $t \to -\infty$.

(b) i. Sketch the vector field for the system $\dot{x} = -y$, $\dot{y} = -x$ by taking a few representative points on the phase plane. Identify the nature of the fixed point at $(0, 0)$.
ii. Obtain an equation for the trajectories of the system.(*Hint : The governing equations above imply* $x\,\dot{x} - y\,\dot{y} = 0$.) Identify the stable and unstable manifolds on the phase plane.
iii. Decouple the system using new variables u and v, where $u = x + y$ and $v = x - y$. Rewrite the (decoupled) system in terms of u and v. Solve for $u(t)$ and $v(t)$ starting from an initial condition (u_0, v_0).
iv. What are the equations for the stable and unstable manifolds in terms of (i) x and y, (ii) u and v.

2+(2+2+2+2)

4. (a) Consider the dynamics of a point (x, y) described by the equations

$$\frac{dx}{dt} = -y \qquad \frac{dy}{dt} = x + ay.$$

where a is a real parameters. Find the stability of the fixed point for $0 < a < 2$.

(b) Is the map

$$\begin{aligned} x_{n+1} &= x_n \cos\theta - (y_n - x_n^2)\sin\theta \\ y_{n+1} &= x_n \sin\theta + (y_n - x_n^2)\cos\theta \quad (0 < \theta < \pi) \end{aligned}$$

area-preserving?

(c) Take the system

$$\begin{aligned} \dot{x} &= x(3 - x - 2y) \\ \dot{y} &= y(2 - y - x). \end{aligned}$$

Find all the fixed points and classify them from the eigenvalues of the stability matrix. Draw schematically, the flow diagram demarkating the basin(s) of attraction. 3+2+(2+3)

5. (a) Consider the map $x_{n+1} = rx_n - x_n^3$ for real values of x. Locate the fixed points and find the range of values of r for which the map has one or more stable fixed point. Find the values of r for which the fixed points are (i) superstable, (ii) marginally stable.

(b) Consider the map $x_{n+1} = -(1+r)x_n - x_n^2 + 2x_n^3$ for real values of x and $-2 \le r \le 2$. Locate the fixed points and analyse their stability. 4+6

6. (a) Generate a fractal with the following prescription: The unit square is divided into N equal squares and out of them M are chosen at random and discarded; this procedure is repeated with every existing square. Is it self-similar in structure? Find the box dimension of this fractal in terms of M and N.

(b) Locate the superstable fixed point for the map $x_{n+1} = x_n^2$.

(c) A map $x_{n+1} = f(x_n)$ has a fixed point (stable or unstable) at $x = x^*$, where the multiplier is $\mu = f'(x^*)$. For an initial value x_0, show that

$$x_n - x^* = (x_0 - x^*)\mu^n$$

under linearisation at all stages. Now consider two close initial points x_0 and x_0'. Show that the n-th members will be separated by

$$x_n' - x_n = (x_0' - x_0)e^{n\lambda}$$

where $\lambda = \log\mu$. Can we infer about the appearance of chaos from the value of μ? 4+2+4

7. (a) Given baker's map in the form for $[0 < X, Y < 1]$ and $[0 < a \le \frac{1}{2}]$

$$\begin{aligned} X_{n+1} &= 2X_n \\ Y_{n+1} &= aY_n \end{aligned}$$

for $0 \le X_n \le \frac{1}{2}$

$$
\begin{aligned}
X_{n+1} &= 2X_n - 1 \\
Y_{n+1} &= aY_n + \frac{1}{2}
\end{aligned}
$$

for $\frac{1}{2} \le X_n \le 1$,
(i) Find the fractal dimension d of the attractor of the baker's map. (ii) Find Liapunov exponent in X and Y directions. (iii) Does chaos or strange attractor appear in this map for $a = 1/2$?

(b) Define the Lyapunov index λ for a map given by $X_{n+1} = f(X_n)$ and hence calculate it for the tent map described by

$$
\begin{aligned}
X_{n+1} &= rX_n \qquad \text{for } 0 \le X_n \le 0.5 \\
&= r(1 - X_n) \quad \text{for } 0.5 \le X_n \le 1
\end{aligned}
$$

for $0 \le r \le 2$. Under which condition is tent map chaotic? (2+3+2)+(1+2)