M. Sc. (Physics) 3rd Semester Examination 2018
PHY 512 (Statistical Mechanics)

Full Marks : 50 Time : 2 hours

Group A

Answer any five

1. For a canonical ensemble prove that the entropy is given by

$$S = -k_B \sum_r p_r \log p_r$$

where p_r is the probability for the occurrance of the r-th state. [5]

2. Starting from Sackur Tetrode equation for entropy of an ideal mono-atomic gas

$$S/k_B = N \ \log\left(\frac{V}{N}\right) + \frac{3N}{2} \ \log\left(\frac{4\pi mE}{3Nh^2}\right) + \frac{5N}{2}$$

Show that the chemical potential is given by $\mu = k_B T \log\left(\frac{N\lambda^3}{V}\right)$ where $\lambda = \frac{h}{\sqrt{2\pi m k_B T}}$ is the thermal wavelength. [5]

3. Consider a system of extreme relativistic gas consisting of N particles moving in one dimension. Show that the partition function is given by

$$Q_N(L,T) = \frac{1}{N!}\left[2L\left(\frac{kT}{hc}\right)\right]^N$$

L being the space available in one dimension. [5]

4. Show that for a system in canonical ensemble

$$\langle(\Delta E)^3\rangle = k^2\left[T^4\left(\frac{\partial C_V}{\partial T}\right)_V + 2T^3 C_V\right]$$

where the symbols have their usual meaning. [5]

5. A system has two energy levels 0 and E. These can be occupied by fermions with temperature T and chemical potential μ. The fermions are non-interacting. Write the partition function and find the average occupation number of the state with energy E. You do not have to consider the spin factor. [2+3]

6. Consider a microcanonical ensemble of N interacting particles where each particle can have two energy levels 0 and ϵ. The state with energy ϵ is doubly degenerate. The total energy of the system is U.

   (a) Find the number of microstates in terms of U, N and ϵ.

(b) Find the temperature and show that

$$U = \frac{2N\epsilon}{\exp(\epsilon/k_BT) + 2}$$

[2+3]

Group B

Answer any five

7. The Hamiltonian of the spin-1 Ising model in 1-dimension(in absence of any external magnetic field) is given by

$$H = -J\sum_{i=1}^{N} S_iS_{i+1}$$

where S_i can take the values -1, 0 and 1.

(a) If $J > 0$, what is the ground state (stable configuration at $T = 0$) of this model ?

(b) Construct the 3×3 transfer matrix of this system. Using periodic boundary condition, show that the canonical partition function can be expressed as $Z_N = \mathrm{Tr}P^N$, P being the transfer matrix.

[1+4]

8. (a) Identify the broken symmetries (if any), and define appropriate order parameters for the following transitions:

   i. gas-liquid transition

   ii. para to ferro transition in Heisenberg ferromagnets

   (b) For a magnetic system in zero external field, the Landau free energy is written as $f(m) = \frac{1}{2}rm^2 + um^4$, m being the order parameter. If $u > 0$ and $r = a(T - T_c)$, show that the system exhibits a second order phase transition at T_c. Why don't we have terms containg m or m^3 in the expansion of $f(m)$?

[2+3]

9. Let the critical exponents β and δ be defined as :
i) $H = 0, t \to 0, M \sim |t|^\beta$
ii) $t = 0, H \to 0, M \sim |H|^{1/\delta}$
Here $t = (T - T_c)/T_c$, M is the order parameter, and H is the conjugate field.
Starting from the static scaling hypothesis $G(\lambda^{a_t}t, \lambda^{a_H}H) = \lambda G(t, H)$, show that $\beta\delta = a_H/a_t$. [5]

10. For a triangular spin-lattice in 2 dimension with nearest-neighbour interaction, the renormalization group recursion relation is given by

$$K' = R(K) = 2K\Big(\frac{e^{3K} + e^{-K}}{e^{3k} + 3e^{-k}}\Big)^2$$

how that it corresponds to 3 fixed points : i) $K^* = 0$ ii) $K^* = \infty$ and iii) $K^* = \frac{1}{4}\ln(2\sqrt{2}+1)$. Hence gue that the system posseses a non-zero, finite critical temperature. [4+1]

a) From the solution of the Langevin equation

$$v(t) = v_0 e^{-\gamma t} + \frac{1}{m}\int_0^t \eta(t_1) e^{-\gamma(t-t_1)} dt_1$$

(Here $\eta(t)$ is a random force and γ is the coefficient of friction) compute the mean squared velocity $\langle v^2(t) \rangle$.

b) Let a system (originally described by the Hamiltonian H) be perturbed by a small potential $V_0 \sin^2 \omega t$. Working in the framework of linear response theory, show that the new partition function can be expressed as $z(H)(1 - \frac{\beta V_0}{2})$, where $z(H)$ is the equilibrium partition function.

[3+2]

a) Consider a non-ideal gas, consisting of 10 particles. In the cluster expansion of the partition function, there is a term :

$$\int f_{1,2} f_{3,4} f_{3,5} f_{8,9} d^3 r_1 \ldots\ldots\ldots\ldots d^3 r_{10}$$

Draw the corresponding graph and find $\{m_l\}$.

b) Define a Markov process. Show that the joint probability distribution function for such a process can be written as :

$$p_n(\xi_n, t_n; \xi_{n-1}, t_{n-1}; \ldots; \xi_1, t_1) = \left[\prod_{j=1}^{n-1} p(\xi_{j+1}, t_{j+1} | \xi_j, t_j)\right] p(\xi_1, t_1)$$

for $n \geq 2$.

[2+3]