2021

MATHEMATICS — HONOURS

First Paper

(Module – I)

Full Marks : 50

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

[Throughout the paper $\mathbb{R}$ and $\mathbb{Z}$ denote the set of real numbers and set of integers respectively]

Group - A

(Marks : 35)

Answer ***any seven*** questions.

1. (a) If p and q are relatively primes, then find out *gcd* of $p+q$ and $p-q$.

(b) Correct or justify the following statement :

For any positive integer n, $f(n) = n^2 + n + 13$ is always a prime integer. 3+2

2. (a) If $a \equiv b \pmod{m}$, then show that $a^n \equiv b^n \pmod{m}$ for all positive integers n. Is the converse true? Justify your answer.

(b) Prove or disprove : $gcd(0, x) = x$ where x is a natural number. (2+1)+2

3. (a) Solve the congruence $12x \equiv 9 \pmod{15}$.

(b) State Euclid's second theorem. 3+2

4. (a) Using Chinese Remainder theorem, solve the linear congruence $9x \equiv 21 \pmod{30}$.

(b) If p is a prime integer and k is any positive integer, prove that

$$\phi\left(p^k\right) = p^k\left(1 - \frac{1}{p}\right),$$

where ϕ denotes the Euler's phi function. 3+2

5. (a) Prove that for any non-zero complex number z, $\arg z - \arg(-z) = \pm\pi$, according as $\arg z > 0$ or < 0, where $\arg z$ is the principal argument of z.

(b) Find all complex numbers z such that $exp(z + \bar{z}) = 1$. 3+2

6. (a) If $\cos h^{-1}(x + iy) + \cos h^{-1}(x - iy) = \cos h^{-1}a$, where x, y, a are real numbers and $a > 1$, prove that the point (x, y) lies on an ellipse.

(b) Find the product of all values of $(-i)^{\frac{1}{4}}$. 3+2

7. (a) If $a_1, a_2, a_3, \ldots, a_n$ are n positive numbers and $a_1 + a_2 + a_3 + \ldots + a_n = S$, then show that

$$\frac{S}{S-a_1} + \frac{S}{S-a_2} + \frac{S}{S-a_3} + \ldots + \frac{S}{S-a_n} \geq \frac{n^2}{n-1}$$

(b) If α is an imaginary root of the equation $x^7 = 1$, then find the value of $(\alpha^6 + 1)(\alpha^5 + 1)(\alpha^4 + 1)(\alpha^3 + 1)(\alpha^2 + 1)(\alpha + 1)$. 3+2

8. Using Sturm's functions, show that the roots of $x^4 + 4x^3 - x^2 - 10x + 3 = 0$ are all real and distinct. 5

9. Solve $x^4 + 3x^3 + 5x^2 + 4x + 2 = 0$ by Ferrari's method. 5

10. Calculate Sturm's functions and find the number and nature of real roots of the equation $x^5 - 5x + 2 = 0$. 3+1+1

11. Show that all the imaginary roots of the equation $x^7 = 1$ are special roots. If α is a special root of $x^7 = 1$, form the equation whose roots are $\alpha + \alpha^6$, $\alpha^2 + \alpha^5$ and $\alpha^3 + \alpha^4$. 2+3

12. (a) If α, β, γ are the roots of the equation $ax^3 + 3bx^2 + 3cx + d = 0$, find the value of

$$(2\alpha - \beta - \gamma)(2\beta - \gamma - \alpha)(2\gamma - \alpha - \beta).$$

(b) Find the value of k, for which the equation $x^4 + 4x^3 - 2x^2 - 12x + k = 0$ has 4 real and unequal roots. 3+2

13. Show that if the roots of the equation $x^4 + x^3 - 4x^2 - 3x + 3 = 0$ are increased by 2, the transformed equation is a reciprocal equation. Solve the reciprocal equation and hence obtain the solution of the given equation. 2+3

Group - B

(Marks : 15)

Answer ***any three*** questions.

14. (a) Prove of disprove : If X, Y and Z are subsets of a set S, then $X \Delta Z = X \Delta Y$ implies $Z = Y$.

(b) Let $A = \{1, 2, 3, 4\}$ and a relation ρ on A is given by $\rho = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 3), (3, 2), (2, 1), (4, 1), (3, 1)\}$. Verify whether the relation is an equivalence relation. 3+2

15. (a) If $f : A \to B$ and $g : B \to C$ are two mappings such that $g \circ f : A \to C$ is surjective.

Verify whether g is surjective. Is it necessary that f is surjective? Justify your answer.

(b) Find two mappings f and g such that $f \circ g \neq g \circ f$. (2+1)+2

16. (a) Prove that a finite semigroup in which both cancellation laws hold is a group.

(b) Does the set $M_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a,b,c,d \in \mathbb{R} \right\}$ form a group with respect to matrix multiplication? Justify your answer. 3+2

17. (a) If each element of a group G is its own inverse, then prove that G is abelian. Is the converse true? Justify your answer.

(b) Give an example of a group (G, o) in which $\text{o}(a) \cdot \text{o}(b) \neq \text{o}\,(a\text{o}b)$, for some $a, b \in G$; where $\text{o}(a)$ means order of the element a in G. (2+1)+2

18. (a) In a group (G, o), $\text{o}(a) = 5$ and $a\text{o}b\text{o}a^{-1} = b^2$. Show that if $b \neq e$ (the identity element of G), then $\text{o}(b) = 31$.

(b) If $(H, \cdot)$ is a subgroup of $(G, \cdot)$, show that $H^{-1} = H$, where $H^{-1} = \{a^{-1} : a \in H\}$. 3+2