2021

MATHEMATICS — HONOURS

Third Paper

(Module – VI)

Full Marks : 50

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Group – A

(Marks : 15)

Answer ***any three*** questions.

1. (a) State Cauchy's general principle of convergence for infinite series of real numbers. Using this principle show that the series $\sum \frac{1}{n}$ is not convergent.

(b) Prove that the series $\sum_{n=1}^{\infty} (-1)^{n-1} \sqrt{n}$ is convergent. Examine its absolute convergence.

(1+2)+(1+1)

2. State Cauchy's condensation test for convergence of series of positive real numbers. Use it to prove that the series $\sum_{n=3}^{\infty} \frac{1}{n \log n (\log \log n)}$ is divergent. 2+3

3. (a) Consider a real valued function f on closed interval $[a, b]$. Are the conditions of Rolle's theorem necessary for existence of a point in (a, b) where $f'(x) = 0$? Justify your answer.

(b) Prove that $\sin 46^\circ \simeq \frac{\sqrt{2}}{2}\left(1 + \frac{\pi}{180}\right)$. 2+3

4. (a) A function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = 0, \text{ if } x = 0 \text{ or } x \text{ is irrational}$$

$$= \frac{1}{q^3}, \text{ if } x = \frac{p}{q}, \text{ where } p \in \mathbb{Z},\ q \in \mathbb{N} \text{ and } gcd\,(p, q) = 1.$$

Show that f is differentiable at $x = 0$ and $f'(0) = 0$.

(b) Show that there does not exist a function ϕ such that $\phi'(x) = f(x)$, where

$$f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & x = 2. \end{cases}$$

3+2

5. One corner of a rectangular sheet of paper is folded over so as to reach the opposite edge (lengthwise) of the sheet. If the area of the folded part be minimum, show that the crease divides the width in the ratio 2 : 3. 5

Group – B

(Marks : 35)

Answer ***any five*** questions.

6. (a) Prove that $\dfrac{1}{x^2y^2}$ is an integrating factor of $\left(x^2y^4 - x\right)dy + \left(x^4y^2 - y\right)dx = 0$.

(b) Find the differential equation that represents all parabolas each of which has a latus rectum $4a$ and whose axes are parallel to the x-axis. 2+5

7. (a) Obtain the orthogonal trajectories of the family of circles $x^2 + y^2 = 2ax$ in polar form.

(b) Find the complete primitive and the singular solution of the differential equation

$$\sin\left(x\frac{dy}{dx}\right)\cos y - \cos\left(x\frac{dy}{dx}\right)\sin y = \frac{dy}{dx}.$$

3+4

8. (a) Find the general solution of $(x+1)^2\dfrac{d^2y}{dx^2} - 2(x+1)\dfrac{dy}{dx} + 2y = 1$, given that $y = x + 1$ and $y = (x + 1)^2$ are linearly independent solutions of the corresponding homogeneous equation.

(b) Verify exactness of the equation $\sin^2 x\dfrac{d^2y}{dx^2} = 2y$. 5+2

9. (a) Find the eigen values and eigen functions for the differential equation $\dfrac{d^2y}{dx^2} + \lambda y = 0$ which satisfies the boundary conditions $y(0) = 0$ and $y(\pi) = 0$.

(b) Prove or disprove : The functions e^x, $\cosh x$, $\sinh x$ are linearly independent. 5+2

10. Solve by the method of undetermined coefficients : $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 4y = 8x^2 + 3 + \cos 2x$. 7

11. (a) Solve the following simultaneous equation :

$$\frac{dx}{dt}+3x+y=e^{t}$$
$$\frac{dy}{dt}-x+y=e^{2t}.$$

(b) Reduce the ordinary differential equation $\frac{d^2y}{dx^2}-\frac{2}{x}\frac{dy}{dx}+\left(a^2+\frac{2}{x^2}\right)y=0$ to normal form. 5+2

12. (a) State the theorem on existence and uniqueness of solution of the initial value problem

$$\frac{dy}{dx}=f(x,y),\ y(x_0)=y_0.$$

(b) Using this theorem, prove that the initial value problem $\frac{dy}{dx}=xy+y^3,\ y(0)=0$ has a solution for

$|x|\le\frac{1}{2}.$ 2+5

13. (a) Prove that $y=x+\frac{1}{x}$ is a solution of $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0$ and hence determine the general solution of the equation.

(b) Solve : $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z.$ 4+3

14. (a) Solve the partial differential equation $\left(p^2+q^2\right)x=pz,\ p\equiv\frac{\partial z}{\partial x},\ q\equiv\frac{\partial z}{\partial y}$ by Charpit's method.

(b) Form the partial differential equation which has the general solution $z=e^{2x+3y}f(2x-3y),$ where f is an arbitrary function. 5+2

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