2022

MATHEMATICS — HONOURS

Paper : CC-2

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Throughout the question the symbols $\mathbb{N}$, $\mathbb{Z}$ denote respectively the set of natural numbers, set of integers. The other symbols have their usual meanings.

1. Choose the correct alternative with proper justification, **1 mark** for correct answer and **1 mark** for justification : 2×10

(a) Number of equivalence relations on the set $\{1, 2, 3\}$ is

(i) 2 (ii) 3 (iii) 4 (iv) 5.

(b) Let $f : \mathbb{Z} \to \mathbb{Z}^+$, $\mathbb{Z}^+$ is the set of non-negative integers, is defined by $f(x) = \frac{1}{2}(x + |x|)$, then

(i) f is injective but not surjective

(ii) f is not injective but surjective

(iii) f is injective and surjective

(iv) f is neither injective nor surjective.

(c) The remainder when $6.7^{32} + 7.9^{45}$ is divided by 4 is

(i) 1 (ii) 2 (iii) 3 (iv) 4.

(d) The principal value of $(-1)^i$ is

(i) e^{π} (ii) $e^{-\pi}$ (iii) $e^{\pi/2}$ (iv) $e^{-\pi/2}$.

(e) If $gcd(a, b) = p$, a prime number, then $gcd(a^{2023}, b)$ is

(i) p (ii) p^{2023} (iii) $2023p$ (iv) p^2.

(f) If the roots of the equation $x^3 - 7x^2 + ax + 2023 = 0$ are integers, then the value of a is

(i) 1 (ii) 289 (iii) -289 (iv) 119.

(g) For positive real numbers a, b and c, the least value of $a^{-1} + b^{-1} + c^{-1}$ subject to the condition $a + b + c = 2023$ is

(i) $\frac{1}{2023}$ (ii) $\frac{9}{2023}$ (iii) $\frac{3}{2023}$ (iv) $\frac{2023}{9}$.

(h) The points $z = x + iy$ on the Argand plane, satisfying $e^{iz} = -1$ lie

(i) in an ellipse (ii) in a straight line (iii) in a circle (iv) in a parabola.

(i) The rank of the matrix $\begin{pmatrix} 1 & n \\ n & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ is

(i) 1, for every n (ii) 2, for every n (iii) 2, except $n = -1$ (iv) 1, except $n = -1$.

(j) A particular solution of the difference equation $u_{x+2} + u_{x+1} + u_x = 2^x$ is

(i) $\frac{2^x}{7}$ (ii) $\frac{2^x}{3} + 4$ (iii) $-\frac{2^x}{7}$ (iv) $\frac{2^x}{3}$.

2. Answer ***any four*** questions :

(a) Find the roots of the equation $z^n = (z+1)^n$, where n is a positive integer > 1. Show that the points which represent them in the z-plane are collinear. 3+2

(b) If $a, b, c, d > 0$ and $a + b + c + d = 1$, prove that

$$\frac{a}{1+b+c+d} + \frac{b}{1+a+c+d} + \frac{c}{1+a+b+d} + \frac{d}{1+a+b+c} \geqslant \frac{4}{7}.$$ 5

(c) If $\sin(\theta + i\varphi) = \tan\beta + i\sec\beta$, prove that $\cos 2\theta \cosh 2\varphi = 3$. 5

(d) Use Sturm's function to show that roots of the equation $x^3 + 3x^2 - 3 = 0$ are real and distinct. 5

(e) Find the values of k, for which the equation $x^4 + 4x^3 - 2x^2 - 12x = k$ has four real and unequal roots. 5

(f) Solve the equation $x^4 + 11x^2 + 10x + 50 = 0$ by Ferrari's method. 5

(g) Solve : $u_n = 7u_{n-1} - 12u_{n-2} + 3^n$ given that $u_0 = 0; u_1 = 2, (n \in \mathbb{N})$. 5

3. Answer ***any four*** questions :

(a) P_1 be a relation defined on the set of integers $\mathbb{Z}$ such that $P_1 = \{(x, y) | x, y \in \mathbb{Z}, x - y = 5n, n \in \mathbb{Z}\}$. Show that P_1 is an equivalence relation. If P_2 be another relation defined as

$$P_2 = \{(x, y) | x, y \in \mathbb{Z}, x - y = 3n, n \in \mathbb{Z}\}$$

show that the relation $P_1 \cup P_2$ is symmetric but not transitive. 3+2

(b) If $f : A \rightarrow B$ be a mapping and P, Q are two non-empty subsets of A, then show that

$$f(P \cup Q) = f(P) \cup f(Q).$$

Give an example to show that $f(P \cap Q) \neq f(P) \cap f(Q)$. 3+2

(c) (i) Consider the set $S = \{1, 2, 3, 4\}$ and the partition $\{\{1\}, \{2\}, \{3,4\}\}$ of S. Find the equivalence relation corresponding to the above partition.

(ii) A function $f : z \to z$ is defined by

$$f(x) = \frac{x}{2}, \text{ if } x \text{ is even}$$
$$= 7, \text{ if } x \text{ is odd}$$

Find a left inverse of f, if it exists. 3+2

(d) If d is the *gcd* of two nonzero integers a and b, prove that there exist two integers u and v such that $d = au + bv$. Are u and v unique? Justify your answer. 3+2

(e) Solve the system of linear congruences by Chinese remainder theorem : $x \equiv 1 \ (mod\ 17)$, $x \equiv 1 \ (mod\ 7)$, $x \equiv 4 \ (mod\ 5)$. 5

(f) If $\leq$ be a relation defined on $\mathbb{N}$ by $a \leq b$ if and only if $|a - b| < 1$, then prove that $\leq$ is an equivalence relation. Is it a partial order relation? Justify your answer. 3+2

(g) (i) Find the general solution, in positive integers, of the equation $12x - 7y = 8$.

(ii) Find the number of integers less than 900 and prime to 900. 4+1

4. Answer ***any one*** question : 5×1

(a) For what values of λ the following system of linear equations is solvable? Then solve it for those values of λ :

$$\begin{aligned} x + y + z &= 2 \\ 2x + y + 3z &= 1 \\ x + 3y + 2z &= 5 \\ 3x - 2y + z &= k \end{aligned}$$

(b) Find the rank of the matrix A, where

$$A = \begin{pmatrix} 1 & 3 & 7 & 1 & 2 \\ 4 & 0 & 5 & 2 & 9 \\ 3 & 3 & 4 & 7 & 4 \\ 0 & 0 & 6 & 6 & -3 \end{pmatrix}$$

by reducing to its row-reduced echelon form.