2021

MATHEMATICS — HONOURS

Paper : DSE-A-1

(Advanced Algebra)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

[*Notations have usual meanings*]

Group – A

(Marks : 20)

1\. Answer ***all*** questions. In each question ***one mark*** is reserved for selecting the correct option and ***one mark*** is reserved for justification. (1+1)×10

(a) Let G be a group of order 22. Then which of the following statements is true?

(i) G is an abelian group.

(ii) G is a simple group.

(iii) G is not a simple group.

(iv) G is a cyclic group.

(b) Let G be a finite group that has only two conjugacy classes. Then which of the following is true?

(i) $|G| = 2$ (ii) $|G| = 4$ (iii) $|G| = 6$ (iv) $|G| = 8$

(c) Which of the following can be a class equation of a group?

(i) 10 = 1+1+1+2+5 (ii) 4 = 1+1+2

(iii) 8 = 1+1+3+3 (iv) 6 = 1+2+3

(d) Let p, q be prime numbers. Then which of the following is true?

(i) Any group of order pq is commutative. (ii) Any group of order pq is simple.

(iii) Any group of order p^2 is commutative. (iv) Any group of order p^2 is simple.

(e) The units of $\mathbb{Z}_6[x]$ are

(i) [1] and [4] (ii) [1] and [5]

(iii) [2] and [5] (iv) [3] and [5].

(f) g.c.d. of $3 + i$ and $-5 + 10i$ in $\mathbb{Z}[i]$ is

(i) $2 + i$ (ii) $-2 + i$ (iii) $2 - i$ (iv) $-2 - i$.

(g) Identify the regular ring.

(i) $\mathbb{Z} \times \mathbb{Z}$ (ii) $\mathbb{Z} \times \mathbb{Q}$ (iii) $\mathbb{Z}_4$ (iv) $\mathbb{Z}_{11}$

(h) Which of the following statements is true for the field $\mathbb{Q}$ of all rational numbers?

(i) $\mathbb{Q}$ has both irreducible element and prime element.

(ii) $\mathbb{Q}$ has irreducible element but does not have prime element.

(iii) $\mathbb{Q}$ has prime element but does not have irreducible element.

(iv) $\mathbb{Q}$ has neither any irreducible element nor any prime element.

(i) Let $f(x) \in \mathbb{Z}[x]$ be a polynomial of degree ≥ 2. Which of the following statements is true?

(i) If $f(x)$ is irreducible in $\mathbb{Z}[x]$, then it is irreducible in $\mathbb{Q}[x]$.

(ii) If $f(x)$ is irreducible in $\mathbb{Q}[x]$, then it is irreducible in $\mathbb{Z}[x]$.

(iii) If $f(x)$ is irreducible in $\mathbb{Z}[x]$, then for all primes p, the reduction $\overline{f(x)}$ of $f(x)$ modulo p is irreducible in $\mathbb{Z}_p[x]$.

(iv) If $f(x)$ is irreducible in $\mathbb{Z}[x]$, then it is irreducible in $\mathbb{R}[x]$.

(j) Which one of the following is not a principal ideal domain?

(i) $\mathbb{Q}[x]$ (ii) $\mathbb{Z}[x]$ (iii) $\mathbb{Z}_5[x]$ (iv) $\mathbb{Z}_{11}[x]$

Group – B

(Marks : 15)

2. Answer ***any three*** questions :

(a) (i) Let K be a Sylow p-subgroup of a finite group G and N be a normal subgroup of G containing K. If K is normal in N, prove that K is normal in G (p is a prime number).

(ii) Let G be a group and S be a G-set. For any $x \in S$, let G_x denote the stabilizer of x. Prove that $G_{bx} = bG_x b^{-1}$ for all $b \in G$ and $x \in S$. 3+2

(b) (i) Prove that no group of order p^2q is simple, where p and q are two distinct prime numbers.

(ii) Show that every group of order 99 has a normal subgroup of order 9. 3+2

(c) (i) Prove that any two Sylow p-subgroups of a finite group are conjugate (where p is a prime number).

(ii) Let G be a finite group and H be a Sylow p-subgroup of G. Prove that H is a unique Sylow p-subgroup of G if and only if H is a normal subgroup of G. 3+2

(d) Prove that a commutative group G is a simple group if and only if G is isomorphic to $\mathbb{Z}_p$ for some prime number p. 5

(e) Let n be a positive integer and H be a subgroup of S_n of index 2. Prove that $H = A_n$. 5

Group – C

(Marks : 30)

3. Answer ***any six*** questions :

(a) (i) Let R be a Euclidean domain with Euclidean valuation δ and $a, b \in R$. If a and b are associates in R, then prove that $\delta(a) = \delta(b)$.

(ii) Let R be a Euclidean domain with Euclidean valuation δ and $a, b \in R$. If $a|b$ and $\delta(a) = \delta(b)$, then show that a and b are associates in R. 3+2

(b) (i) Show that $1+\sqrt{-5}$ is irreducible in the ring $\mathbb{Z}\left[\sqrt{-5}\right]$.

(ii) Show that 2 is not prime in the ring $\mathbb{Z}\left[\sqrt{5}\right]$. 3+2

(c) (i) Prove that the polynomial $x^6 + x^3 + 1$ is irreducible over $\mathbb{Q}$.

(ii) Find the quotient field of the integral domain $\mathbb{Z}[i]$. 3+2

(d) Prove that every principal ideal domain is a unique factorization domain. Is the converse true? Justify your answer. 2+3

(e) Let $M_2(\mathbb{R})$ denote the set of all 2×2 matrices over $\mathbb{R}$. Show that the ring $M_2(\mathbb{R})$ with respect to usual addition and multiplication of matrices is a regular ring. 5

(f) (i) Prove that the Polynomial ring $\mathbb{Z}_8[x]$ contains infinitely many unit elements.

(ii) Find a monic associate of $3x^5 - 4x^2 + 1$ in the ring $\mathbb{Z}_5[x]$. 3+2

(g) (i) Find $\gcd\left(x^4 + 3x^3 + 2x + 4,\ x^2 - 1\right)$ in $\mathbb{Z}_5[x]$.

(ii) Show that $x^3 + a$ is reducible in $\mathbb{Z}_3[x]$ for each $a \in \mathbb{Z}_3$. 3+2

(h) Prove that in a commutative regular ring with unity, every prime ideal is maximal. 5

(i) Prove that any ring can be embedded in a ring with unity. 5

(j) (i) Prove that in a polynomial ring over a unique factorization domain, product of two primitive polynomials is again primitive.

(ii) In the polynomial ring $\mathbb{Z}[x]$, prove that the polynomial $3x^5 + 10x^4 - 25x^3 + 15x^2 + 20x + 35$ is irreducible. 3+2