18/1/19

2018

PHYSICS

Paper : PHY-413

(Quantum Mechanics-I)

Full Marks : 50

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

Answer ***any five*** questions.

1. (a) A spin-1/2 particle is in the spin state

$$|\beta\rangle = A\begin{pmatrix}1+i\\1-i\end{pmatrix}$$

written in the basis of eigenvectors of $\hat{S}_z$. Find the probabilities of getting $+\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$ if you measure $\hat{S}_x$.

(b) Consider a charged particle of charge q and mass m in a harmonic potential $\frac{1}{2}m\omega^2x^2$ in one dimension in a weak electric field ε along the x direction. Determine the exact energy values. Next, consider the weak field as a perturbation and compute the correction to the energy values using up to second order perturbation theory. 4+(2+4)

2. (a) Consider a particle in a spherical well such that

$$\begin{aligned} V(r) &= 0;\ r \geq R \\ &= -V_0;\ r < R \end{aligned}$$

Show that for the state with zero angular momentum, a bound state is possible only if $V_0 \geq \frac{\pi^2\hbar^2}{8mR^2}$.

(b) The ${}^2P_{1/2}$ and ${}^2S_{1/2}$ levels (for $n \geq 2$, n being the principal quantum number) for the hydrogen atom remain degenerate after considering relativistic correction and spin orbit interactions. Into how many levels will they split in a weak magnetic field B in the z direction? Find the spacing between the lines obtained as a result of the removal of degeneracy. Derive the formula you have to use. 5+5

3. (a) Consider a particle of mass m moving in a double delta potential $V(x) = -\alpha\left[\delta(x+a)+\delta(x-a)\right]$, where α and a are positive constants. Argue that energy eigenstates will have definite parity. Let the energy eigenvalue be $-|E|$. Defining $\mathcal{K} = \sqrt{2m|E|}/\hbar$, prove that k statisfies $\tanh(\mathcal{K}a) = \frac{2m\alpha}{\hbar^2\mathcal{K}} - 1$ for the case of the solution with even parity.

(b) Prove that $(1/\sqrt{2})\,(1 + i\hat{\sigma}_x)$ acting on a two component wavefunction (a column matrix with two rows) can be regarded as the matrix representation of rotation operator about x-axis by an angle $\pi/2$. (2+4)+4

4. (a) If $|n\rangle$ is the n-th eigenstate of the Hamiltonian of a particle of mass m moving under a one-dimensional potential $V(x) = \frac{1}{2}m\omega^2x^2$, calculate (i) $\sqrt{\langle n|\Delta x^2|n\rangle}$ and (ii) $\sqrt{\langle n|\Delta p^2|n\rangle}$ for position ($\hat{x}$) and momentum ($\hat{p}$) operators respectively. Show that the product of (i) and (ii) is minimum for $n = 0$. For any operator $\hat{O}, \Delta\hat{O}^2 \equiv \hat{O}^2 - \langle\hat{O}\rangle^2$. You may find the following definition of annihilation operator $\hat{a}$ handy in your calculation : $\hat{a} \equiv (m\omega\hat{x} + i\hat{p})/\sqrt{2m\omega\hbar}$.

(b) Evaluate the commutator $\left[\hat{x}_\alpha, \exp\left(-i\frac{\hat{\vec{p}}.\vec{a}}{\hbar}\right)\right]$. $\hat{x}_\alpha$ and $\hat{p}_\alpha$ are the components of position and momentum operators respectively. a_αs are dimensionful numbers. Hence, show that for any state $|\psi\rangle, \langle\psi|\hat{x}_i|\psi\rangle = \langle\psi'|\hat{x}_i|\psi'\rangle - a_i$, where, $|\psi'\rangle = \exp\left(-i\frac{\hat{\vec{p}}.\vec{a}}{\hbar}\right)|\psi\rangle$. (5+1)+(3+1)

5. (a) Calculate $\langle 0|\hat{x}(t)\,\hat{x}(0)|0\rangle$. Here, $\hat{x}(0)$ and $\hat{x}(t)$ are position operators at time $t = 0$ and time t respectively for a particle moving in one dimension under $V(x) = \frac{1}{2}m\omega^2x^2$. $|0\rangle$ is the ground state of the Hamiltonian operator.

(b) Motion of a particle with rest mass m and charge q in a uniform magnetic field $\vec{B} = B_0\,\hat{z}$ is described by a Hamiltonian $\hat{H} = -\left(\frac{qB_0}{2mc}\hbar\right)\hat{\sigma}_z$. Initially, the particle is known to be in the state $0.6|+\rangle + 0.8|-\rangle$, where $|\pm\rangle$ are the eigenvectors of $\hat{\sigma}_z$ with eigenvalues ± 1 respectively.

(i) Calculate the probability for finding the state in the $(1/\sqrt{2})(|+\rangle + |-\rangle)$ state as a function of time.

(ii) Calculate the expectation value of $\hat{\sigma}_x$ as a function of time. 4+(3+3)

6. (a) A particle of mass m moves in a potential given by

$$V(r) = c\ln\left(\frac{r}{r_0}\right)$$

where c is a constant. Show that

(i) All eigenstates have the same mean squared velocity. Find this value.

(ii) The spacing between any two levels is independent of mass.

(b) Find the upper bound on the ground state energy E_g for the one dimensional harmonic oscillator using a trial wavefunction of the form

$$\psi(x) = \frac{A}{x^2 + b^2}$$

where A is determined by the normalization and b is an adjustable parameter. You may use the following integral $\int_0^{\pi/2} \sin^{m-1}\theta \cos^{n-1}\theta \, d\theta = \frac{1}{2}B\left(\frac{m}{2}, \frac{n}{2}\right)$ (3+2)+5

7. (a) Interaction Hamiltonian of two spin-$\frac{1}{2}$ particles is given by $\hat{H} = \xi\left(\hat{\sigma}_x^1\hat{\sigma}_x^2 + \hat{\sigma}_y^1\hat{\sigma}_y^2\right)$ where ξ is a constant. What are the energy eigenvalues? What are their degeneracies?

(b) Consider eight identical noninteracting spin-1/2 particles in a three dimensional isotropic harmonic potential of the form $\frac{1}{2}m\omega^2 r^2$. Determine the ground state energy.

(c) The Hamiltonian of a two-level system is given by

$$\hat{H} = \begin{pmatrix} E_1 & A \\ A & E_2 \end{pmatrix} = H_0(A=0) + \hat{H}', \ (E_1, E_2 >> A; E_1 \neq E_2).$$

Obtain the first and the second order correction to E_1 using the stationary state perturbation theory. (4+1)+2+3