2021

MATHEMATICS — HONOURS

Fourth Paper

(Module - VII)

Full Marks : 50

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

$\mathbb{R}$ and $\mathbb{Q}$ denote the set of all real numbers and all rational numbers respectively.

Group - A

(Marks : 30)

Answer ***any six*** questions.

1. Correct or Justify the following :

(a) The set $A = \{(x, y) : x, y$ both are irrational$\}$ is neither open nor closed set in $\mathbb{R}^2$.

(b) The function $f : [-1, 1] \to \mathbb{R}$ defined by $f(x) = \begin{cases} -1, & -1 \le x < 0 \\ 0, & x = 0 \\ 1, & 0 < x \le 1 \end{cases}$ is derivative of a function defined on $[-1, 1]$. 2+3

2. Let $\lim\limits_{(x,y)\to(a,b)} f(x, y) = A \in \mathbb{R}$. Prove that $\lim\limits_{x\to a} \lim\limits_{y\to b} f(x, y) = A = \lim\limits_{y\to b} \lim\limits_{x\to a} f(x, y)$ if $\lim\limits_{x\to a} f(x, y)$ exists for each y in a neighbourhood of b and $\lim\limits_{y\to b} f(x, y)$ exists for each x in a neighbourhood of a. 5

3. Let $f(x, y) = \begin{cases} \dfrac{x^3 + y^3}{x + y}, & x + y \ne 0 \\ 0, & x + y = 0 \end{cases}$.

Examine if the function f is continuous at (0, 0). Find the repeated limits if they exist. 3+2

4. Let $f(x, y) = \begin{cases} \dfrac{xy^3}{x^2 + y^2}, & (x, y) \ne (0,0) \\ 0, & (x, y) = (0,0) \end{cases}$.

Show that $f_{yx}(0,0) \ne f_{xy}(0,0)$. Which condition of Schwarz's theorem is not satisfied by f? 3+2

Please Turn Over

5. What do you mean by an implicit function $y = \phi(x)$ defined by $F(x, y) = 0$ near (a, b)? Verify implicit function theorem for $x^2 + xy + y^2 - 1 = 0$ near $(0, -1)$. 2+3

6. If $u = \dfrac{x}{(1-r^2)^{1/2}}, v = \dfrac{y}{(1-r^2)^{1/2}}, w = \dfrac{z}{(1-r^2)^{1/2}}$, where $r^2 = x^2 + y^2 + z^2$,

show that $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = (1-r^2)^{-5/2}$. 5

7. If $x^2 = vw,\ y^2 = wu,\ z^2 = uv$ and $f(x, y, z) = \phi(u, v, w)$, show that $xf_x + yf_y + zf_z = u\phi_u + v\phi_v + w\phi_w$. 5

8. If $u = xy$, $v = z - 2x$, and $w = f(u, v) = 0$ where w is a differentiable function of u & v such that $\dfrac{\partial f}{\partial v} \neq 0$ and if it is given that z is dependent on the two independent variables x & y, prove that $x\dfrac{\partial z}{\partial x} - y\dfrac{\partial z}{\partial y} = 2x$.

5

9. Show by using method of Jacobian the functions $u = x + y - z,\ v = x - y + z,\ w = x^2 + y^2 + z^2 - 2yz$ are not independent. Find the relation among them. 5

10. Find the points on the ellipse $5x^2 - 6xy + 5y^2 = 4$ at which the tangent is at the greatest distance from the origin. 5

11. Show that there exists θ with $0 < \theta < 1$ such that

$\sin x.\sin y = xy - \dfrac{1}{6}\left[\left(x^3 + 3xy^2\right)\cos\theta x.\sin\theta y + (y^3 + 3x^2 y)\sin\theta x.\cos\theta y\right]$. 5

Group - B

(Marks : 20)

Answer ***any four*** questions. 5×4

12. Find the asymptotes of the curve $x(x^2 - y^2) + y(3y - x) = 0$. Also find the line containing the points of intersection of the above curve and all its asymptotes.

13. Show that for the curve $x = a + b\log\left(b + \sqrt{b^2 - y^2}\right) - \sqrt{b^2 - y^2}$, sum of the sub-normal and sub-tangent is constant.

14. Find the equation of the evolute of the astroid $(x-1)^{2/3} + (y-1)^{2/3} = a^{2/3}\ (a > 0)$.

15. Find the length of the perimeter of the cardioid $r = a(1 - \cos\theta)$ and show that the arc of the upper half of the curve is bisected by $\theta = \frac{2\pi}{3}$.

16. Find the area of the surface generated by revolving the cardioid $r = a(1 + \cos\theta)$ about the initial line.

17. Find the position and nature of the double points of the curve $y(y - 6) = x^2(x - 2)^3 - 9$.

18. Find the moment of inertia of the elliptic plate of mass m and bounded by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about its major axis.