2022

MATHEMATICS — HONOURS

Paper : DSE-B-2

(Advanced Mechanics)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Symbols and Notations have their usual meanings unless otherwise stated.

Group – A

1. Answer the following multiple choice questions with only one correct option. Choose the correct option and justify : 2×10

(a) Two particles are connected by a rigid weightless rod of constant length. The degrees of freedom of this two-particle system is

(i) 3 (ii) 4

(iii) 5 (iv) 6.

(b) The Lagrangian for a simple pendulum of length l and mass of the bob m with angle of deflection θ, is given by

(i) $L=\frac{1}{2}ml^2\dot{\theta}^2-mgl(1-\cos\theta)$ (ii) $L=\frac{1}{2}ml\dot{\theta}+mgl(1-\cos\theta)$

(iii) $L=\frac{1}{2}ml^2\dot{\theta}^2-mgl(1+\cos\theta)$ (iv) $L=\frac{1}{2}ml\dot{\theta}-mgl(1-\cos\theta)$.

(c) The expression for kinetic energy (T) of a particle of mass m in cylindrical polar coordinate system (r, θ, z) is given by

(i) $T=\frac{1}{2}m\left(\dot{r}^2+r\dot{\theta}^2+\dot{z}^2\right)$ (ii) $T=\frac{1}{2}m\left(\dot{r}^2+\dot{\theta}^2+r^2\dot{z}^2\right)$

(iii) $T=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2+\dot{z}^2\right)$ (iv) $T=\frac{1}{2}m\left(\dot{r}^2+\dot{\theta}^2+r\dot{z}^2\right)$.

(d) The transformation of Lagrangian to Hamiltonian is done by

(i) Legendre transformation (ii) Lorentz transformation

(iii) Galilean transformation (iv) Laplace transformation.

(e) The Hamiltonian of a plane pendulum of length 'l' and mass of the bob 'm' is given by

(i) $\frac{1}{2}ml^2\dot{\theta} + mgl\cos\theta$ (ii) $\frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta$

(iii) $\frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$ (iv) $\frac{1}{2}ml^2\dot{\theta} - mgl\cos\theta$.

(f) The action of the path of a physical system is given by

(i) $\int_{t_0}^{t_1} L(q_i, \dot{p}_i, t)\,dt$ (ii) $\int_{t_0}^{t_1} L(q_i, \dot{q}_i, t)\,dt$

(iii) $\int_{t_0}^{t_1} L(p_i, \dot{q}_i, t)\,dt$ (iv) $\int_{t_0}^{t_1} L(p_i, \dot{p}_i, t)\,dt$.

(g) The Lagrangian of a system in two dimensions is given in cartesian coordinates (x, y) as

$$L = \frac{1}{2}m\dot{x}^2 + m\dot{x}\dot{y}.$$

If p_x and p_y are the momentum conjugates corresponding to the coordinates x and y, respectively, then the Hamiltonian H of the system is given by

(i) $H = \frac{p_x p_y}{m} + \frac{p_y^2}{2m}$ (ii) $H = \frac{p_x p_y}{m} + \frac{p_x^2}{2m}$

(iii) $H = \frac{p_x p_y}{m} - \frac{p_y^2}{2m}$ (iv) $H = \frac{p_x p_y}{m} - \frac{p_x^2}{2m}$.

(h) If the kinetic energy T does not depend on the generalized velocities $\dot{q}_j$, the generalized force Q_j of the system with Lagrangian L is given by

(i) $Q_j = \frac{\partial L}{\partial \dot{q}_j}$ (ii) $Q_j = \frac{\partial T}{\partial \dot{q}_j}$

(iii) $Q_j = \frac{\partial L}{\partial q_j}$ (iv) $Q_j = \frac{\partial T}{\partial q_j}$.

(i) In Hamilton's principle, the action integral $I = \int_{t_1}^{t_2} L\, dt$ is stationary for arbitrary variations of the path in the

(i) Phase space
(ii) n-dimensional space R^n
(iii) Configuration space
(iv) State space.

(j) Let H be the Hamiltonian of a given dynamical system with one degree of freedom for which $[p, H] = 0$, q being coordinate and p the conjugate momentum. Then

(i) p is constant
(ii) q is constant
(iii) both p and q are constants
(iv) p and q are both variables.

Group – B

Unit - 1

(Marks : 10)

2. Answer ***any two*** questions :

(a) What are constraints? Give an example of a non-holonomic constraint system.

Show that $\sum_{j=1}^{n} \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L$ is a constant of motion for a conservative, scleronomic system. 2+1+2

(b) If L is the Lagrangian for a system with n degrees of freedom, show by direct substitution that

$$L' = L + \frac{dF(q_1, q_2, \ldots, q_n, t)}{dt},$$

also satisfies Lagrange's equations, where F is an arbitrary, differentiable function of its arguments. 5

(c) For a system described by generalised coordinates $q_1, q_2, \ldots, q_n$; define generalised momentum p_i corresponding to the generalised coordinate q_i. Establish the relation $\dot{p}_i = \frac{\partial L}{\partial q_i}, i = 1, 2, \ldots, n$.

What are velocity-dependent potentials? 1+2+2

(d) A particle of mass m is projected with an initial velocity u at an angle α to the horizontal. Choosing a suitable coordinate system, write down the Lagrangian of the system and deduce the equations of motion. Integrate these with the given initial conditions. 2+3

Unit - 2

(Marks : 15)

3. Answer *any three* questions :

(a) State the Hamilton's principle and derive Lagrange's equations of motion from it. 5

(b) In spherical polar coordinates (r, θ, ϕ), the Lagrangian of a particle of mass m under the influence of a central potential $V(r)$ is given by

$$L\left(r, \dot{r}, \theta, \dot{\theta}, \dot{\phi}\right) = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right) - V(r).$$

Find the momentum conjugate to the cyclic coordinate. Also, find the Routhian and write down the Routhian equations of motion. 2+1+2

(c) (i) The Lagrangian of a system with two degrees of freedom is given by

$$L = m\dot{x}\dot{y} + m\left(x^2 + y^2\right).$$

Write down the expression for the canonical momentum p_r conjugate to r in polar coordinates.

(ii) The Lagrangian of a system is given by $L = \frac{1}{2}ml^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right) - mgl\cos\theta$,

where m, g, l are constants. Show that $\dot{\phi}\sin^2\theta$ is conserved. 3+2

(d) If for a certain mechanical system $H = p^2q^2 - \lambda pq$, where λ is a real constant, then show that pq is a constant of motion. Obtain the Hamilton's equations of motion for a simple pendulum. 2+3

(e) Define action of a mechanical system. Show that if a coordinate corresponding to a rotation is cyclic, the system remains invariant under such a coordinate rotation and angular momentum is conserved. 2+3

Unit - 3

(Marks : 10)

4. Answer *any two* questions :

(a) The Lagrangian of a one-dimensional anharmonic oscillator with unit mass is given by

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega^2x^2 - \alpha x^3 + \beta x\dot{x}^2.$$

Find the Hamiltonian and show that the canonical equations of motion are

$$\dot{x} = \frac{p}{1+2\beta x},\ \dot{p} = \beta\dot{x}^2 - \omega^2 x - 3\alpha x^2.$$ 2+3

(b) A particle of mass m and coordinate q has the Lagrangian $L = \frac{1}{2}m\dot{q}^2 - \frac{\lambda}{2}q\dot{q}^2$,

where λ is a constant. Find the Hamiltonian and deduce Hamilton's equation of motion. 3+2

(c) State and prove the principle of stationary action. Obtain the modified Hamilton's principle from the principle of stationary action. 3+2

(d) A particle of unit mass is projected so that its total energy is E in a field it is moving in and the potential energy is $V(r)$ at a distance r from the origin. Using the principle of stationary action, show that the differential equation of the path is $c^2\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right] = r^4\left[E - V(r)\right]$, where c is an arbitrary constant. 5

Unit - 4

(Marks : 10)

5. Answer ***any two*** questions :

(a) What is a canonical transformation? Show that the transformation

$$P = \frac{1}{2}\left(p^2 + q^2\right)$$

$$Q = \tan^{-1}\left(\frac{q}{p}\right)$$

is canonical. 2+3

(b) Show that the generating function for the transformation $p = \frac{1}{Q}$, $q = PQ^2$ is $F = \frac{q}{Q}$. 5

(c) Derive the Hamilton-Jacobi equation for Hamilton's principal function S. Solve the Hamilton-Jacobi equation for the system whose Hamiltonian is given by $H = \frac{p^2}{2} - \frac{\mu}{q}$. 3+2

(d) Define Poisson bracket. If $F(p, q, t)$ and $G(p, q, t)$ are two constants of motion then show that the Poisson bracket $[F, G]$ is also a constant of motion. 1+4
