2021

MATHEMATICS — HONOURS

Paper : DSE-B(2)-3

(Advanced Mechanics)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Symbols and Notations have their usual meanings.

Group - A

1. Answer the following multiple choice questions with only one correct option. Choose the correct option and justify : 2×10

(a) A constraint of the form :

$$\vec{f}\left(\vec{r}_j, t\right) = 0,\ j = 1, 2, \ldots, N$$

is called a

(i) kinematical constraint (ii) bilateral constraint

(iii) unilateral constraint (iv) geometric constraint.

(b) The number of degrees of freedom of a rigid body is

(i) 9 (ii) 3

(iii) 6 (iv) None of these.

(c) If the Hamiltonian of a one-dimensional dynamical system is given by

$$H = \frac{1}{2}\left(\alpha p^2 + \beta q^2 + 2\gamma pq\right),$$

where α, β, γ are constants, then the Poisson bracket $\{p, \mathrm{H}\}$ is equal to

(i) $-\alpha q + \gamma p$ (ii) $\beta q + \gamma p$

(iii) $-\alpha q - \gamma p$ (iv) $-\beta q - \gamma p$.

(d) Two weightless, inextensible rods AB and BC are suspended at A and jointed by a flexible joint at B. The degrees of freedom of the system is

(i) 3 (ii) 4

(iii) 5 (iv) 6.

(e) The Lagrangian of a free particle of mass m and velocity v is given by

(i) $L = \frac{mv}{2}$ (ii) $L = -\frac{mv^2}{2}$

(iii) $L = \frac{mv^2}{2}$ (iv) $L = -\frac{mv}{2}$.

(f) The Hamiltonian of a particle of mass m is $H = \frac{p^2}{2m} + pq$, where q is the generalized coordinate and p is the corresponding momentum. The Lagrangian of the particle is

(i) $\frac{m}{2}(\dot{q} + q)^2$ (ii) $\frac{m}{2}(\dot{q} - q)^2$

(iii) $\frac{m}{2}(\dot{q}^2 + q\dot{q} - q^2)$ (iv) $\frac{m}{2}(\dot{q}^2 - q\dot{q} + q^2)$.

(g) Consider the Hamiltonian (H) and the Lagrangian (L) for a free particle of mass m and velocity v. Then

(i) H and L are independent of each other

(ii) H and L are related but have different dependence on v

(iii) H and L are equal

(iv) H is a quadratic in v but L is not.

(h) 'If a symmetry is found to exist in a dynamical problem, then there is a corresponding constant of motion.'— This is

(i) Noether's theorem (ii) Fermat's principle

(iii) Liouville's theorem (iv) None of these.

(i) A dynamical system having kinetic energy T and potential energy V is described by the Hamiltonian H. Assume that the equations defining the generalized coordinates do not depend on time t. Then,

(i) $H = T + V$ and is conserved

(ii) $H = T + V$ but it is not conserved

(iii) $T + V$ is conserved but is not equal to H

(iv) none of these.

(j) A mechanical system is described by the Hamiltonian $H(q, p) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$. As a result of canonical transformation generated by $F(q, Q) = -\frac{Q}{q}$, the Hamiltonian in the new coordinate Q and new momentum P becomes

(i) $\frac{1}{2m}Q^2P^2 + \frac{1}{2}m\omega^2Q^2$ (ii) $\frac{1}{2m}Q^2P^2 + \frac{1}{2}m\omega^2P^2$

(iii) $\frac{1}{2m}P^2 + \frac{1}{2}m\omega^2Q^2$ (iv) $\frac{1}{2m}Q^2P^4 + \frac{1}{2}m\omega^2P^{-2}$.

Group - B

Unit - 1

2. Answer ***any two*** questions : 5×2

(a) What is degree of freedom? Define holonomic and scleronomic constraints. Write the type of constraint(s) for motion of a body on an inclined plane under gravity. 1+2+2

(b) What do you understand by cyclic coordinates? Show that the generalized momentum corresponding to a cyclic coordinate is a constant of motion. 2+3

(c) A double pendulum consists of a mass m_1 attached to one end a massless rod of length l_1 whose other end is hinged, and a second mass m_2 attached to another massless rod of length l_2 whose other end is hinged to the mass m_1. Using suitable generalized coordinates, setup the Lagrangian and write the Lagrange's equations for the system. 5

(d) What are dissipative forces? What is Rayleigh's dissipation function and how is it related with the rate of energy dissipation of a system? Use Rayleigh's dissipation function to write Lagrange's equations of motion for a dissipative system where the nonpotential forces are linear in the generalized velocities. 1+2+2

Unit - 2

3. Answer ***any three*** questions : 5×3

(a) In a dynamical system, the kinetic and potential energies are $T = \frac{1}{2}\frac{\dot{q}_1^2}{a + bq_2^2} + \frac{1}{2}\dot{q}_2^2$,

$$V = C + dq_2^2 \text{ where } a, b, c, d \text{ are constants.}$$

Determine $q_1(t)$ and $q_2(t)$ by Routh's process of ignoration of coordinates. 5

(b) A certain oscillator with generalized coordinate q has Lagrangian

$$L = \frac{1}{2}\dot{q}^2 - \frac{1}{2}q^2 .$$

Verify that $q^* = \sin t$ is a motion of the oscillator, and show directly that it makes the action integral stationary in any time interval $[0, \tau]$. 5

(c) What is Legendre transformation? Using Legendre transformation, obtain Hamilton's equation of motion from Lagrange's equation of motion. 2+3

(d) State Hamilton's principle. Derive Hamilton's principle from D'Alembert's principle. 1+4

(e) (i) The Lagrangian of a system with two degrees of freedom is given by $L = \frac{1}{2} m \dot{x}^2 + m \dot{x} \dot{y}$. Write down the Hamiltonian of the system.

(ii) Suppose that the equations defining generalized coordinates q_i for a system with n degrees of freedom do not involve time. Show that in this case $\sum_{i=1}^{n} p_i \dot{q}_i = 2T$, where p_i are the momenta conjugate to q_i and T is the kinetic energy of the system. 3+2

Unit - 3

4. Answer ***any two*** questions : 5×2

(a) Find Hamilton's equations in spherical polar coordinates for a particle of mass m moving in three dimensions in a force field of potential V. 5

(b) Explain the principle of stationary action. How does this principle lead to Fermat's principle? 3+2

(c) A spherical pendulum consists of a mass m attached to one end of a massless rod of length l which is hinged at the other end and is free to oscillate in any vertical plane. Using suitable generalized coordinates, write down the Hamiltonian and find the equations of motion for the coordinates. 5

(d) What are Poincaré integral invariants and Poincaré–Cartan integral invariants? What is the difference between the two? 2+2+1

Unit - 4

5. Answer ***any two*** questions : 5×2

(a) Show that the transformation

$$P = q \cot p$$
$$Q = \log\left(\frac{\sin p}{q}\right)$$

is canonical. Show also that the generating function for this transformation is

$$F = e^{-Q}\left(1 - e^{2Q}\right)^{\frac{1}{2}} + q \sin^{-1} q \, e^{Q}$$ 3+2

(b) Show that the function $S = \frac{m\omega}{2}\left(q^2 + \alpha^2\right)\cot \omega t - m\omega q \alpha \operatorname{cosec} \omega t$ is a solution of the Hamilton–Jacobi equation for Hamilton's principal function for the linear harmonic oscillator with

$$H = \frac{1}{2m}\left(p^2 + m^2 \omega^2 q^2\right).$$

Show that this function generates a correct solution to the motion of the harmonic oscillator. 3+2

(c) Define the Poisson bracket of two dynamical variables $u = u\,(q_i, p_i, t)$ and $v = v\,(q_i, p_i, t)$ where q_i, p_i are the canonical variables and t is time.

Show that Poisson brackets are canonical invariants. 2+3

(d) State and prove Liouville's theorem for a Hamiltonian system. 1+4