2021

MATHEMATICS — HONOURS

Paper : CC-12

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

1. Choose the correct answer and justify (1 mark for right answer and 1 mark for justification) : 2×10

(a) Largest order among the elements of $Z_{30} \times Z_{20}$ is

(i) 30 (ii) 20 (iii) 60 (iv) 10

(b) Let G be a group of order 77 and a be an element of G of order 7. The number of conjugates of a is :

(i) 1 (ii) 7 (iii) 6 (iv) 77

(c) Let G be a cyclic group of order 40. Then which one of the following is true?

(i) $G \simeq \mathbb{Z}_2 \times \mathbb{Z}_{20}$ (ii) $G \simeq \mathbb{Z}_4 \times \mathbb{Z}_{10}$ (iii) $G \simeq \mathbb{Z}_8 \times \mathbb{Z}_5$ (iv) $G \simeq \mathbb{Z}_{20} \times \mathbb{Z}_2$

(d) Number of non-isomorphic abelian groups of order $(2017)^3$ is

(i) 1 (ii) 2017 (iii) 3 (iv) 3×2017

(e) Number of automorphisms on $\mathbb{Z}_2 \times \mathbb{Z}_2$ is

(i) 1 (ii) 6 (iii) 4 (iv) 8

(f) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear functional defined by $T(a, b) = (2a + b, a - 3b)$ and T^* be the adjoint of T, then $T^*(3, 5)$ is equal to

(i) $(6, -5)$ (ii) $(11, -12)$ (iii) $(0, 0)$ (iv) none of these

(g) Let $\{u_1, u_2, u_3, u_4, u_5\}$ be an orthogonal basis of $\mathbb{R}^5$, y be a vector in $\mathbb{R}^5$, W_1 = Span $\{u_1, u_2\}$, W_2 = Span $\{u_3, u_4, u_5\}$. If $W^{\perp}$ denotes the orthogonal complement of W, then which of the following is false?

(i) $W_1 = W_2^{\perp}$

(ii) $W_2 = W_1^{\perp}$

(iii) there are two vectors Z_1 in W_1 and Z_2 in W_2 such that $y = Z_1 + Z_2$

(iv) y is orthogonal to W_1 and to W_2

(h) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that $T(1, 2) = (2, 3)$, $T(0, 1) = (1, 4)$. Then $T(5, 6)$ is

(i) $(6, -1)$ (ii) $(-6, 1)$ (iii) $(-1, 6)$ (iv) $(1, -6)$

(i) Let A be a matrix of the quadratic form $(x_1 + 2x_2 + ... + nx_n)^2$, then the sum of the entries of A is :

(i) $\sum n$ (ii) $\sum n^2$ (iii) $\sum n^3$ (iv) $\frac{n(n+1)}{2}$

(j) If the quadratic form $x^2 + \lambda(y^2 + z^2) + 2xy$ is positive definite, then

(i) $\lambda = 5$ (ii) $\lambda > 1$ (iii) $\lambda < 1$ (iv) none of these.

Unit – I

(Group Theory)

2. Answer ***any four*** questions :

(a) (i) Let G be a finite group and 'f' be a automorphism of G such that for all $a \in G$, $f(a) = a$ if and only if $a = e$. Show that for all $g \in G$ there exists $a \in G$ such that $g = a^{-1} f(a)$.

(ii) If G is a non-commutative group, then prove that G has a non-trivial automorphism. 3+2

(b) Show that S_3 has a trivial centre and it can not be expressed as an internal direct product of two non-trivial subgroups. 2+3

(c) Suppose that G is a finite abelian group and G has no element of order 2. Show that the mapping $f : G \to G$ defined by $f(g) = g^2$ for all $g \in G$ is an automorphism of G. Show by an example, that if G is infinite the mapping need not be automorphism. 3+2

(d) (i) Show that $Z(\text{Aut}(G)) = \{e\}$ if for a group G, $Z(G) = \{e\}$.

(ii) Prove that $R^* \simeq R^+ \times Z_2$ where R^* is the set of all non-zero real numbers and R^+ is the set of all positive real numbers. 3+2

(e) (i) If an abelian group G is the internal direct product of its subgroups H and K, then prove that $H \simeq G/K$ and $K \simeq G/H$.

(ii) Show that the Klein 4-group is isomorphic to the direct product of a cyclic group of order 2 with itself. 3+2

(f) (i) If $Z(G)$ be the centre of a group G, then prove that $G/Z(G) \simeq Inn(G)$

(ii) Exhibit an automorphism of Z_6 that is not an inner automorphism. 3+2

(g) (i) State fundamental theorem of finite abelian groups.

(ii) Find all abelian groups (up to isomorphism) of order 360. 2+3

Unit – II

(Linear Algebra)

3. Answer ***any five*** questions :

(a) Diagonalise the matrix $A = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & 0 \end{pmatrix}$ orthogonally. 5

(b) Find a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that the set of all vectors (x_1, x_2, x_3) satisfying the equation $4x_1 - 3x_2 + x_3 = 0$ is ker T. 5

(c) (i) Reduce the equation $2x^2 + 5y^2 + 10z^2 + 4xy + 6xz + 12yz$ into its canonical form.

(ii) Find all possible Jordan canonical forms for the matrix whose characteristic polynomial is $(t-2)^4 (t-5)^3$ and minimal polynomial is $(t-2)^2 (t-5)^3$. 3+2

(d) Let $V = \mathbb{R}^3$ be the vector space over $\mathbb{R}$ and V^* be its dual space.
Let $f_1, f_2, f_3 \in V^*$ such that $f_1(x, y, z) = x - 2y$, $f_2(x, y, z) = x + y + z$, $f_3(x, y, z) = y - 3z$.
Prove that $\{f_1, f_2, f_3\}$ is a basis for V^* and find a basis for V for which it is the dual basis. 5

(e) Obtain the eigenvalues, eigenvectors and eigenspaces of the matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. 5

(f) Show that $\left\langle \sum_j a_j x^j, \sum_k b_k x^k \right\rangle = \sum_{j,k} \frac{a_j b_k}{j+k+1}$ defines an inner product on the space $\mathbb{R}[x]$ of polynomials over the field $\mathbb{R}$. 5

(g) Define Annihilator of a subspace.

If $W = \{(x, y, z) : x - 2y - 3z = 0\}$ be a subspace of $\mathbb{R}^3$. Find the Annihilator of W. 2+3

(h) For the matrix $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$, find an orthogonal matrix P such that $P^t AP$ is a diagonal matrix. 5