2022

PHYSICS

Module : PHY-423

(Statistical Mechanics)

Full Marks : 50

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Answer ***any five*** questions.

1. (a) Find the number of micro-states available to a monatomic ideal gas with number of particles N, volume V and energy lying between E and $E + dE$. Hence obtain an expression for its entropy.

(b) Starting from Bose-Einstein distribution show that the amount of energy carried by photons with frequency between ω and $\omega + d\omega$ is

$$E(\omega)d\omega = \frac{Vh}{2\pi^3 c^3} \times \frac{\omega^3 d\omega}{e^{\frac{h\omega}{2\pi k_B T}} - 1}$$

5+5

2. (a) The chemical potential of a boson is zero or negative. Justify.

(b) Explain why a degenerate Fermi gas exerts pressure whereas a Bose does not.

(c) Show that the density fluctuation in Grand Canonical Ensemble may be written as

$$\frac{\overline{(\Delta n)^2}}{\bar{n}^2} = -\frac{k_B T}{V}\frac{1}{v}\left(\frac{\partial v}{\partial P}\right) = \frac{k_B T}{V}\chi T$$

where n is the density, v is specific volume, χT is the isothermal compressibility and the other symbols have their usual meaning. 3+2+5

3. Consider N-number of interacting gas particles with potential $U(r)$. Let, $f(r) = e^{-\beta U(r)} - 1$.

(a) Show that the pressure may be written as $P = \frac{NkT}{V}[1 + V\alpha]$ where, in the leading order of $f(r)$,

$$\alpha = -\frac{N}{2V^2}\int f(r)d^3r$$

(b) Assume $f(r) = \begin{cases} -1, & \text{for } r < r_0 \\ -\left(1-e^{-\beta U(r)}\right), & \text{for } r > r_0 \end{cases}$ Show that, to the leading order in $f(r)$, this potential reproduces van der Waals' gas equation.

5+5

4. (a) Consider a Bose gas in 2 dimension with surface area A. Derive an expression for the number of particles in the excited state (N_e) and in the ground state N_0 as a function of $y\left(=e^{\mu/kT}\right)$, T and A. Show that the system does not exhibit BE condensation unless $T \to 0$. You may use

$$\int_0^\infty \frac{dx}{y^{-1}e^x - 1} = -ln(1-y).$$

(b) If the Gibbs free energy of a system follows the generalized homogeneous form $G\left(\lambda^{a_t} t, \lambda^{a_H} H\right) = \lambda G(t, H)$, show that

$$\beta = \frac{1-a_H}{a_t}$$

both for $T > T_c$ and $T < T_c$.

The critical exponent β is defined as : $H = 0, t \to 0, M \sim |t|^\beta$. Here $t = (T - T_c)/T_c$ (T_c being the critical temperature), M is the order parameter, and H is the conjugate field. (3+2)+5

5. (a) Find the critical exponent corresponding to the thermodynamic function

$$f(t) = t^{-\frac{1}{3}} + 2t^{-\frac{1}{6}}$$

Here $t = (T - T_c)/T_c$, which measures the deviation of the temperature T from the critical value T_c.

(b) Consider 3 Ising spins ($S_i = \pm 1$) that form an equilateral triangle. The Hamiltonian is

$$\mathcal{H} = -J \sum_{i,j} S_i S_j$$

Construct the partition function (i) directly, by considering all possible spin configurations, and (ii) using the transfer matrix method. Show that these two are equal. 2+(3+4+1)

6. (a) For a magnetic system in zero external field, the Landau free energy is written as

$$F(m) = \frac{1}{2} rm^2 + um^4$$

m being the order parameter. If $u > 0$ and $r = a(T - T_C)$, find the ground state of the system both for $T > T_c$ and $T < T_c$. Argue that there is a phase transition of second order at T_c. Why there should not be any term containing m or m^3 in the expansion of $F(m)$?

(b) For a particular spin model with nearest-neighbour interaction energy J_1 and second-nearest neighbour interaction energy J_2, the renormalization group recursion relations are found to be :

$K' = 2K^2 + L$

$L' = K^2$.

Here $K = J_1/k_B T$ and $L = J_2/k_B T$.

Show that it corresponds to 3 fixed points : (i) $(K^*, L^*) = (0, 0)$ (ii) $(K^*, L^*) = (\infty, \infty)$ and (iii) $(K^*, L^*) = (1/3, 1/9)$. (4+2+1)+3

7. (a) Let a system in equilibrium (originally described by the Hamiltonian H) be perturbed by a small potential V. The system is characterized by a physical variable D. The equilibrium expectation value of D is $\langle D \rangle_{eq}$, while its value under the influence of the perturbed Hamiltonian is $\langle D \rangle_{pert}$. Working in the framework of linear response theory, show how $\langle D \rangle_{pert}$ is connected to $\langle D \rangle_{eq}$. What would happen if V and D are completely uncorrelated?

(b) From the solution of the Langevin equation

$$v(t) = v_0 e^{-\gamma t} + \frac{1}{m}\int_0^t \eta(t_1) e^{-\gamma(t-t_1)} dt_1$$

(Here $\eta(t)$ is a random force and γ is the coefficient of friction) compute $\left\langle v^2(t) \right\rangle$. Show that absence of dissipation in the system will lead to an unphysical system. (4+1)+(3+2)