M.Sc. (Physics) 4th Semester 2021
PHY 523 (Nonlinear Dynamics)

Full marks: 50 Time: 2.5 hrs

(*2 hours for answering and 30 minutes for downloading, scanning, and mailing back*)

Answer any *five* questions.

Instructions:

(a) Write your complete *Examination Roll Number*, (with college code and subject category) and *Registration Number* (from an earlier admit card) at the top of your answer script. They should be clearly legible.

(b) Do not write your name or class roll number anywhere.

(c) Write page number on top of each page.

(d) Scan the complete answer script into *a single pdf file* and mail it to the e-mail from where you got this question paper. The return mail should preferably be sent from the e-mail to which the question paper was delivered, in "reply" mode.

(e) The answer script file for the paper PHYAAA(where AAA is the paper code like 521, 522, and so on)must be named as instructed:
Note that your Examination Roll Number is of the form ZZZ/PHY/XXXXXX, where ZZZ is the college identifier (like C91, 031, etc.), and XXXXXX is a 6-digit number like 191099.

— For CU students, the filename for the paper PHYAAA must be CUXXXXXXPHYAAA.pdf.
For example, the script of PHY521 coming from C91/PHS/191099 must be named
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— For students of Lady Brabourne College, the name should be
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1. (a) Using linear stability analysis, classify the fixed point(s) for the system

$$\dot{x} = \frac{x}{5} - \frac{x}{3+x}$$

(b) Sketch qualitatively the vector fields for $\dot{x} = r\,x - 4\,x^3$, as the parameter r is varied. Identify the type of bifurcation and sketch the bifurcation diagram. For cases where the origin is the only fixed point, estimate the rate of decay towards the origin.

(c) Find the nature of the fixed point at the origin and plot nearby phase trajectories for the linear system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

3+(2+1+2)+2

2. (a) Explain the terms (i) globally stable, (ii) Liapunov stable and (iii) asymptotically stable in reference to a fixed point $\mathbf{x}^*$ of a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. Illustrate each by means of an example.

(b) For the system $\ddot{x} + 2\,\epsilon\,\dot{x} + x = 0$, where $\epsilon << 1$,

i. identify the fixed point(s) and its(their) nature.

ii. make a schematic phase plot for the system in the vicinity of the fixed point(s).

iii. obtain the exact solution $x(t)$, for the initial conditions $x(0) = 0$ and $\dot{x}(0) = 1$.

4+(2+1+3)

3. (a) State the Poincare-Bendixson theorem in *your own words.*

(b) Express the system

$$\begin{aligned} \dot{x} &= -y + x(1 - x^2 - y^2) \\ \dot{y} &= x + y(1 - x^2 - y^2)\ , \end{aligned}$$

in polar coordinates. Show, using schematic figures where needed, that trajectories are trapped in a region $0.5 < r < 2$, where $r^2 = x^2 + y^2$.

(c) The van der Pol oscillator is described by

$$\ddot{x} + \mu\,(x^2 - 1)\,\dot{x} + x = 0\,.$$

i. In the strong nonlinear limit, $\mu >> 1$, make a suitable choice of variables to identify the nullclines of the system and plot them on a phase plot.

ii. On the same phase plot as obtained above, plot a typical trajectory of the system. Identify the path(s) corresponding to different time scales.

iii. Estimate the time period of oscillation, clearly showing the intermediate calculations.

2+3+(1+1+3)

4. (a) Consider the system given by

$$\begin{aligned}
\dot{x} &= \mu\, x - y - (x^2 + y^2)\, x \\
\dot{y} &= x + \mu\, y - (x^2 + y^2)\, y,
\end{aligned}$$

where μ is a parameter. Linearize the system and find the Jacobian matrix at the origin. Show that a Hopf bifurcation results as μ is varied.

(b) Express the above system in polar coordinates and show the different phase plots as μ is changed.

(c) For the map $x_{n+1} = 3.42 x_n(1 - x_n)$, how many fixed points or cycles will exist, and what will be their nature? 3+3+4

5. (a) Generate a fractal called Sierpinski triangle, with the following prescription: Start with an equilateral triangle S_0 of unit area. Divide it into four smaller congruent equilateral triangles and remove the central triangle to get S_1. Repeat the last step with each of the remaining smaller triangles. What will be the area of the set S_k, and what will be the similarity dimension for the limiting set S_∞?

(b) Consider the map $x_{n+1} = a \sin x_n$. Locate the fixed points of this map and analyse their stability for (i) $a = \pi/6$ and (ii) $a = \pi/3$. 4+6

6. (a) For the map

$$\begin{aligned}
x_{n+1} &= a y_n + x_n - x_n^2 \\
y_{n+1} &= x_n
\end{aligned}$$

find the fixed points and analyse their stability for (i) $a = \frac{1}{3}$ and (ii) $a = -\frac{1}{8}$.

(b) Consider the synchronisation of fireflies with external light. Write the equation for the phase of the fireflies. Derive the nature of synchronisation when the natural frequency of fireflies and the frequency of the external light are (i) equal, and (ii) unequal. 6+4

7. (a) Consider the dynamics of a point (x, y) described by the equations

$$\frac{dx}{dt} = x^2 - y^2, \qquad \frac{dy}{dt} = xy - 4.$$

Locate the fixed points and analyse their stability.

(b) Explain briefly how is the phenomenon of chemical oscillation consistent with the second law of thermodynamics.

(c) In the space spanned by the variables (X, Y, Z) consider the set of points enclosed in a volume V_0 at time $t = 0$. Each point evolves according to the equations

$$\begin{aligned}
\dot{X} &= -X + YZ \\
\dot{Y} &= -Y + ZX \\
\dot{Z} &= 1 - XY.
\end{aligned}$$

Find the volume $V(t)$ which the points will occupy at time t. 5+2+3