## 2019

## PHYSICS

Paper: PHY-4 13
(Quantum Mechanics-I)
Full Marks : 50
The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

Answer any five questions.

1. (a) An operator A , corresponding to a physical quantity $\alpha$, has two normalized eigenfunctions $\psi_{1}(x)$ and $\psi_{2}(x)$, with eigenvalues $\alpha_{1}$ and $\alpha_{2}$. An operator $B$, corresponding to another physical quantity $\beta$, has normalized eigenfunctions $\phi_{1}(x)$ and $\phi_{2}(x)$, with eigenvalues $\beta_{1}$ and $\beta_{2}$. The eigenfunctions are related via

$$
\begin{aligned}
\psi_{1} & =\left(2 \phi_{1}+3 \phi_{2}\right) / \sqrt{13} \\
\psi_{2} & =\left(3 \phi_{1}-2 \phi_{2}\right) / \sqrt{13}
\end{aligned}
$$

$\beta$ is measured on some state and the value $\beta_{1}$ is obtained. If $\alpha$ is then measured and then $\beta$ again, find the probability of obtaining $\beta_{1}$ a second time.
(b) In a harmonic oscillator in three dimensions, the energy eigenvalues can be written as $(n+3 / 2) \hbar \omega$ where $n=n_{1}+n_{2}+n_{3}$ and the corresponding eigenfunction can be written as $\psi_{n_{1}, n_{2}, n_{3}}(x, y, z)$. Determine how the wave function changes under parity operator for a given value of $n$. You can use the knowledge of the eigenfunctions for the one dimensional harmonic oscillator.
(c) Determine the degeneracy of a 2-dimensional harmonic oscillator for the $n$th eigenstate with energy $\mathrm{E}_{n}=(n+1) \hbar \omega$.
(d) Use the virial theorem to determine the expectation value of the kinetic energy in a 3-dimensional harmonic oscillator.
$4+2+2+2$
2. (a) Consider the infinite square well potential in 1 dimension with a delta function hump in the middle such that $V(x)=\infty$ for $x>a$ and $x<-a$, and $V(x)=\lambda \delta(x)$ otherwise.
What are the conditions to be satisfied by the wave function in such a potential? Solve the problem exactly to find the energy eigenvalues. Solution may be in terms of a transcendental equation.
(b) Explain whether the value of the matrix element $\langle 200| p_{z}|200\rangle$ will be zero without doing actual calculation. Here, $|n, t, m\rangle$ is an energy eigenstate of the hydrogen atom.

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(c) Consider a correction to the potential energy of the hydrogen atom in the form

$$
V^{\prime}(r)=V(r)+\frac{1}{6} r_{0}^{2} \nabla^{2} V(r)+\ldots
$$

where $V(r)$ is the electrostatic potential. If the second term is treated as a perturba correction to the energy for the 2 P level of hydrogen atom in first order?
3. (a) Consider the potential $V(x)=a_{0}|x|$. Do you expect the eigenfunctions to have Justify.
(b) For the above potential, apply the variational method using a Gaussian trial wave fu an estimate of the ground state energy.
(c) Suppose a photon is in a state $|\psi\rangle$. Consider the operator $P_{\psi}=|\psi\rangle\langle\psi|$. Argue that $P \psi$ is hermitian.
(d) Show that the expectation value of a physical quantity represented by the $\langle Q\rangle=\operatorname{Tr}\left(P_{\psi} Q\right)$.
4. (a) For two identical spin $1 / 2$ particles, compute $\mathbf{s}_{1} \cdot \mathbf{s}_{2}$ for the spin triplet and singlet The Hamiltonian for the two particles is given by $H=H_{1}\left(r_{l}\right)+H_{2}\left(r_{2}\right)+V(\mid r$ $H_{1}\left(H_{2}\right)$ is the Hamiltonian for the first (second) particle in absence of the other, identical in form.

Show that the total energy can be written as $E_{1}+E_{2}+\widetilde{E} \pm \widetilde{J}$ where $E_{i}$ are eigenva and $\widetilde{E}$ and $\widetilde{J}$ are integrals involving $V$, and the $\pm$ signs are for the singlet and trip rewrite the energy in terms of $\mathbf{s}_{\mathbf{1}} \cdot \mathbf{s}_{\mathbf{2}}$.
(b) Consider a particle in a spherical well such that

$$
\begin{aligned}
V(r) & =0 ; r>R \\
& =-V_{0} ; r<R
\end{aligned}
$$

Show that a bound state is possible only if $V_{0} \geq \frac{\pi^{2} \hbar^{2}}{8 m R^{2}}$ for $l=0$.
5. (a) If we write the angular momentum operators in spherical polar co-ordinates, we

$$
\begin{aligned}
& L_{y}=i \hbar\left(-\cos \phi \frac{\partial}{\partial \theta}+\cot \theta \sin \phi \frac{\partial}{\partial \phi}\right) \\
& L_{z}=-i \hbar \frac{\partial}{\partial \phi} .
\end{aligned}
$$

Applying the commutator of these operators on an arbitrary function of $\theta$ and $\phi$, show that $L_{x}=i \hbar\left(\sin \phi \frac{\partial}{\partial \theta}+\cot \theta \cos \phi \frac{\partial}{\partial \phi}\right)$.
(b) Calculate the Clebsch-Gordan coefficients for $j_{1}=1$ and $j_{2}=1 / 2$.
(c) Work out the commutation relations $\left[\vec{L} \cdot \vec{S}, L^{2}\right]$ and $\left[\vec{L} \cdot \vec{S}, J_{i}\right]$.
6. The Co-ordinate representation of the momentum operator $(\underline{p})$ for a 1-dimensional system is

$$
\underline{p}=-i \hbar \frac{d}{d x} .
$$

(a) Which of the following statements is correct?

Statement 1: $\langle x| \underline{p}|\psi\rangle=\langle x|-i \hbar \frac{d}{d x}|\psi\rangle$
Statement $2:\langle x| \underline{p}|\psi\rangle=-i \hbar \frac{d}{d x}\langle x \mid \psi\rangle$
(b) Show that, for a 1-dimensional "box" with periodic boundary conditions, the momentum operator is
hermitian.
(c) What is the most general boundary condition that will make the momentum operator hermitian for the 1 -dimensional "box"?

$$
2+4+4
$$

7. The $n^{\text {th }}$ excited state of a simple harmonic oscillator is given, in terms of the ground state $|0\rangle$, by the
relation

$$
|n\rangle=\frac{\left(a^{\dagger}\right)^{n}}{\sqrt{n!}}|0\rangle
$$

(a) Show that these states are properly normalized, i.e., $\langle n \mid n\rangle=1$.
(b) Show that $\langle n| a^{\dagger} a|n\rangle=n$.
(c) Starting from the definition of the ground state, $a|0\rangle=|\Omega\rangle$ where $|\Omega\rangle$ is the null vector, and using the expression $a \propto p-i m \omega x$, find the ground state wave function in the co-ordinate representation.

