2021

MATHEMATICS — HONOURS

Second Paper

(Module - IV)

Full Marks : 50

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Symbols and notations have their usual meanings.

Group - A

[Linear Algebra - I]

(Marks : 35)

Answer ***question no.*** **1** and ***any three*** questions from the rest.

1. Answer ***any one*** of the following questions :

(a) Let a, b, c, d be unequal real numbers.
Show that :

$$\begin{vmatrix} a & -b & -a & b \\ b & a & -b & -a \\ c & -d & c & -d \\ d & c & d & c \end{vmatrix}$$

is never zero. Is the result valid when a, b, c, d are complex numbers? 5

(b) (i) If A and B are square matrices of the same order, does the equality $(A + B)(A - B) = A^2 - B^2$ hold good? Give reason.

(ii) Let A be a complex $n \times n$ matrix. Prove that $A\bar{A}^t = 0 \Rightarrow A = 0$, where 0 is the $n \times n$ null matrix. 2+3

2. (a) Prove that if A and B are two matrices such that $AB = A$ and $BA = B$, then A^t and B^t are idempotent.

(b) If $A = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}$, show that $A^3 - 6A - 9I_3 = 0$. Hence obtain a matrix B such that $BA = I_3$.

(c) Prove that if A is a skew-symmetric matrix then adj A is symmetric or skew-symmetric according as the order of A is odd or even. 3+4+3

3. (a) Suppose $S = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ be a subset of a finite dimensional vector space V over F and $\beta \in V$ be a vector such that $\beta = c_1\alpha_1 + c_2\alpha_2 + ... + c_n\alpha_n$ $c_i \in F$ with $c_1 \neq 0$. Prove that S and $\{\beta, \alpha_2, \alpha_3, ..., \alpha_n\}$ generate the same subspace of V.

(b) Let $W = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 - x_2 + 2x_3 = 0\}$. Show that W is a subspace of $\mathbb{R}^3$. Find a basis of W. 5+5

4. (a) Reduce the matrix $\begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & 0 \end{pmatrix}$ to a row-echelon matrix. Hence solve the system of linear equations

$$\begin{aligned} 2x - 2y \quad &= 12 \\ -2x + y - 2z &= 0 \\ -2y \quad &= 8 \end{aligned}$$

(b) Show that the quadratic form $xy + yz + zx$ is indefinite; also find its rank, index and signature. (3+2)+5

5. (a) Find the values of k so that the following expression forms an inner product over $\mathbb{R}^2$

$(\vec{a}, \vec{b}) = a_1b_1 - 3a_1b_2 - 3a_2b_1 + ka_2b_2$, where $\vec{a} = (a_1, a_2)$, $\vec{b} = (b_1, b_2)$.

(b) Construct an orthonormal set of vectors from a given set of vectors $\vec{u}_1 = (1,0,3,-1)$, $\vec{u}_2 = (1,2,0,1)$, $\vec{u}_3 = (1,1,0,0)$ which generates the same subspace. 5+5

6. (a) If $A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 2 \end{pmatrix}$, find non-singular matrices P and Q such that PAQ is fully reduced normal form.

(b) Prove that the eigenvalues of a 3×3 real symmetric matrix are real.

(c) If λ be an eigenvalue of an orthogonal matrix, then show that $1/\lambda$ is also an eigenvalue of the matrix. 4+3+3

Group - B

[Vector Calculus - I]

(Marks : 15)

Answer ***any three*** questions.

7. (a) Let the vector function $\vec{\alpha}(t)$ have constant direction. Prove that $\vec{\alpha} \times \dfrac{d\vec{\alpha}}{dt} = \vec{0}$.

(b) If $\vec{r} = 3t\hat{i} + 3t^2\hat{j} + 2t^3\hat{k}$, find the value of $\left[\dfrac{d\vec{r}}{dt}\ \dfrac{d^2\vec{r}}{dt^2}\ \dfrac{d^3\vec{r}}{dt^3}\right]$. 3+2

8. (a) Determine the constant 'a', so that the vector $\vec{A} = (x+3y)\hat{i} + (y-2z)\hat{j} + (x+az)\hat{k}$ becomes solenoidal.

(b) Show that the vector $\dfrac{\vec{r}}{r^3}$ is both solenoidal and irrotational, where $r = |\vec{r}|$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

2+3

9. Prove that $\vec{\nabla} \times (\phi\vec{A}) = \phi(\vec{\nabla} \times \vec{A}) + (\vec{\nabla}\phi) \times \vec{A}$, for any scalar function ϕ and vector function $\vec{A}$. 5

10. Find the maximum value of the directional derivative of $\phi = x^2 + z^2 - y^2$ at the point (1, 3, 2). Find also the direction in which it occurs. 5

11. If $f(r)$ is differentiable then prove that $\nabla^2(f(r)) = f''(r) + \dfrac{2}{r}f'(r)$. Hence deduce that $\nabla^2\left(\dfrac{1}{r}\right) = 0$.

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