2019

PHYSICS

Module : PHY-422

(Quantum Mechanics-II)

Full Marks : 50

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Symbols have their usual meanings.

Answer ***any five*** questions.

1. (a) A particle which is initially ($t = 0$) in the ground state of an infinite one dimensional potential box with walls at $x = 0$ to $x = a$ is subjected to a perturbation $V(t) = x^2 \exp(-t/\tau)$ [τ is a constant] during the time interval $0 \leq t \leq \infty$. Calculate up to first order, probability of finding the particle in its first excited state at some time t. [For a particle in an infinite one dimensional potential well the energy eigenvalue E_n and wavefunction $\psi_n(x)$ are as follows : $E_n = \left(n^2\pi^2\hbar^2/2ma^2\right)$ and $\psi_n(x) = (2/a)^{1/2}\sin\left(n\pi x/a\right)$; where n is the quantum number and m is the mass of the particle]

   (b) A particle is in the ground state of a one dimensional harmonic oscillator potential. At $t = 0$, a perturbation $V(x, t) = V_0 x^3 \exp(-t/\tau)$ is turned on, where $x = \left(\hbar/2m\omega\right)^{1/2}\left(\hat{a} + \hat{a}^\dagger\right)$ with $\hat{a}$ and $\hat{a}^\dagger$ are annihilation and creation operators. Calculate up to first order, probability that, after a sufficiently long time ($t \to \infty$), the system will have made a transition to any excited state. 5+5

2. (a) Consider a system of three non-interacting identical spin $\frac{1}{2}$ particles that are in the same spin state $\left|\frac{1}{2}, \frac{1}{2}\right\rangle$ and confined to move in a one dimensional infinite potential well of length a; $V(x) = 0$ for $0 < x < a$ and $V(x) = \infty$ for other values of x. Determine the energy and wave function of the ground state, first excited state and second excited state.

   (b) A proton of energy E is incident on a nucleus of charge Ze. Using the WKB approximation, estimate the transmission coefficient associated with the penetration of the proton inside the nucleus. 5+5

3. (a) In case of elastic scattering between two non-relativistic spinless particles, show that the total scattered wave function is

$$\psi(\mathbf{r}) = \psi_{inc}(\mathbf{r}) - \frac{\mu}{2\pi\hbar^2}\int \frac{e^{ik|r-r'|}}{|r-r'|} V(r')\psi(\mathbf{r}')d^3r'$$

where, μ is the reduced mass and the interaction between the particles depends only on their relative distance.

(given, $\oint \frac{f(z)}{z-z_0}dz = 2\pi i f(z_0)$ where $f(z)$ is an analytic function of z.

(b) Using Born approximation, find the differential scattering cross-section for the Yukawa potential

$$V(r) = V_0 \frac{e^{-\lambda r}}{r}$$

where λ is a constant. Obtain the Rutherford cross-section fomula as a limit of the above result.

(c) Calculate the total cross-section. 4+3+3

4. (a) For a spherically symmetric scattering potential, using partial wave analysis, establish that the total elastic scattering cross-section is given by

$$\sigma_{el} = \frac{4\pi}{k^2}\sum_l (2l+1)\sin^2\delta_l$$

(b) Find the s-wave phase shift, as a function of wave number k, for a spherically symmetric potential which is infinitely repulsive inside a radius r_0 and vanishes outside r_0.

(c) Under what condition can a transformation be regarded as a symmetry? If the time evolution operator $\mathcal{U}(t, t_0)$ is defined as $|\psi(t)\rangle = \mathcal{U}(t,t_0)|\psi(t_0)\rangle$, express $\mathcal{U}(t, t_0)$ in terms of the Hamiltonian operator. 3+3+4

5. (a) The sign operator ($\wedge$) is defined as $\wedge = H_f \Big/ \left(c^2p^2 + m_0^2c^4\right)^{1/2}$, where H_f is the free Dirac Hamiltonian and p is the momentum.

The projection operator ($\wedge_\pm$) is defined as $(\wedge\pm) = \frac{1}{2}(1 \pm \wedge)$ with properties as follows :

$\wedge_+ \Psi_+ = \Psi_+$; $\wedge_- \Psi_+ = 0$; $\wedge_+ \Psi_- = 0$; $\wedge_- \Psi_- = \Psi_-$ where Ψ_+ and Ψ_- are wavefunctions corresponding to positive and negative energy states respectively.

Now any operator can be written as $A = [A] + \{A\}$, where $[A]$ and $\{A\}$ are the even and odd parts of the operator A respectively. Show that $[A] = (1/2)(A + \wedge A \wedge)$ and $\{A\} = (1/2)(A - \wedge A \wedge)$. Hence show that H_f and p are even operators.

If α and β are respective matrices of the Dirac Hamiltonian, show that the even part of β is

$$[\beta] = \left(m_0c^2\right)\wedge \Big/ \left(c^2p^2 + m_0^2c^4\right)^{1/2}.$$

Hence write down the even part of α i.e. $[\alpha]$.

(b) Consider the matrix γ_5, defined as $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$.

Show that (i) $\gamma_5^{\dagger} = \gamma_5$, (ii) $\gamma_5^{2} = \mathbb{1}$, (iii) $\gamma^\mu\gamma_5 + \gamma_5\gamma^\mu = 0$ for all μ. (2+1+2+1)+(1+1+2)

6. (a) Starting from the formula $D(R)T_q^k D^\dagger(R) = \sum_{q'=-k}^{k} D_{qq'}^k(R)T_{q'}^k$, symbols having their usual meanings, find $\left[J_z, T_q^k\right]$ and $\left[J_\pm, T_q^k\right]$.

(b) Define an anti-linear operator. Prove that for a Hamiltonian invariant under time reversal operation, the energy eigenfunctions can be taken to be real in the non-degenerate case. 4+(2+4)

7. (a) Transformation of the Dirac spinor under Lorentz transformation is $\psi'(x') = S(\Lambda)\psi(x)$ where Λ_ν^μ is the Lorentz transformation matrix, $x'^\mu = \Lambda_\nu^\mu x^\nu$. $\Lambda_\nu^\mu \gamma^\nu = S^{-1}(\Lambda)\gamma^\mu S(\Lambda)$ is the condition on S to keep Dirac equation, $\left(i\partial_\mu \gamma^\mu - m\right)\psi = 0$, covariant under Lorentz transformation. Show that

(i) $S^{-1} = \gamma^0 S^\dagger \gamma^0$

(ii) $\bar{\psi}' = \bar{\psi} S^{-1}$

(iii) $\bar{\psi}\gamma^\mu\psi$ is a Lorentz four vector.

(b) Derive the conjugate form of Dirac equation. Show that $j^\mu = \bar{\psi}\gamma^\mu\psi$ is the conserved current, $\partial_\mu j^\mu = 0$. (2+2+2)+(2+2)
