2022

MATHEMATICS — HONOURS

Paper : CC-13

(Metric Space and Complex Analysis)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

[$\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ denote the set of all natural, real and complex numbers respectively.]

(Notations and symbols have their usual meanings.)

1. Answer ***all*** the following multiple choice questions. For each question **1** mark for choosing correct option and **1** mark for justification. 2×10

(a) Which one of the following is not a metric on $\mathbb{R}$?

(i) $d(x, y) = |x - y|$ (ii) $d(x, y) = |x^2 - y^2|$

(iii) $d(x, y) = |x^3 - y^3|$ (iv) $d(x, y) = \dfrac{|x^3 - y^3|}{1 + |x^3 - y^3|}$.

(b) Let (X, d) be a metric space and $A, B \subseteq X$. Choose the correct statement.

(i) $(A \cap B)^{\circ} = A^{\circ} \cap B^{\circ}$ (ii) $\overline{A \cap B} = \bar{A} \cap \bar{B}$

(iii) $A \cap \partial A = \phi$ (iv) $d(A \cup B) = d(A) + d(B)$.

[$\partial(A)$ denotes boundary of A, $d(A)$ denotes diameter of A.]

(c) Let (X, d_1) and (Y, d_2) be two metric spaces and $f : X \to Y$ be a continuous mapping. Then

(i) $f(\bar{A}) = \overline{f(A)}$ for all $A \subseteq X$

(ii) $f(F)$ is closed when $F \subseteq X$ is closed in X

(iii) $f^{-1}(F)$ is closed in X whenever F is closed in Y

(iv) $f^{-1}(\bar{B}) \subseteq \overline{f^{-1}(B)}$ for any $B \subseteq Y$.

(d) Let $A = \left\{ \frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{N} \right\} \cup \{0\}$. Then A as subspace of the real line $\mathbb{R}$ is

(i) compact but not complete (ii) complete but not compact

(iii) compact and complete (iv) neither compact nor complete.

(e) Let $(\mathbb{R}, d_u)$ denotes the usual metric space of $\mathbb{R}$ and $(\mathbb{R}, d^*)$ denotes the discrete metric space of $\mathbb{R}$. Then which of the following statement is true.

(i) d_u and d^* are equivalent metrices (ii) every d_u-open set is d^*-open

(iii) $d_u = d^*$ (iv) $\{0\}$ is open in both metric spaces.

(f) Let $f(z) = \frac{|z|}{\mathrm{Re}(z)}$, if $\mathrm{Re}(z) \neq 0$

$= 0,$ if $\mathrm{Re}(z) = 0$, then

(i) f is continuous everywhere (ii) f is differentiable everywhere

(iii) f is continuous nowhere (iv) f is not continuous at $z = 0$.

(g) The radius of convergence of the power series $\sum \left(\frac{n\sqrt{2} + i}{1 + 2in} \right) z^n$ is

(i) 1 (ii) $\sqrt{2}$

(iii) $\frac{1}{\sqrt{2}}$ (iv) 2.

(h) The Mobius transformation which maps $z_1 = 0$, $z_2 = -i$ and $z_3 = -1$ into $w_1 = i$, $w_2 = 1$, $w_3 = 0$ respectively is given by

(i) $w = -i\left(\frac{z+1}{z-1}\right)$ (ii) $w = i\left(\frac{z+1}{z-1}\right)$

(iii) $w = -i\left(\frac{z-1}{z+1}\right)$ (iv) $w = i\left(\frac{z-1}{z+1}\right)$.

(i) $\int_C \frac{z+4}{z^2 + 2z + 5} dz$, where C is the circle $|z + 1| = 1$ is

(i) $2\pi i$ (ii) $4\pi i$

(iii) 0 (iv) πi.

(j) $\frac{1}{2\pi i}\int_C \frac{z^2+5}{z-3}\,dz$, where C is $|z| = 4$ is

(i) 8 (ii) 10

(iii) 12 (iv) 14.

Unit – 1

(Metric Space)

Answer ***any five*** questions.

2. Suppose that (X_i, d_i) is a metric space for $i = 1, 2, \ldots m$ and let $X = \prod_{i=1}^{m} X_i$. Show that $d : X \to \mathbb{R}$ defined by $d(x, y) = \sum_{i=1}^{m} \frac{1}{i}\left(d_i\left(x_i, y_i\right)\right)$, for all $x = \left(x_1, x_2, \ldots, x_m\right)$ and $y = \left(y_1, y_2, \ldots, y_m\right) \in X$ is a metric on X. 5

3. Let (X, d) be a metric space and Y be a subspace of X. Let Z be a subset of Y. Prove that Z is closed in Y if and only if there exists a closed set $F \subseteq X$ such that $Z = F \cap Y$. 5

4. (a) Let $X = \left\{(x, y) \in \mathbb{R}^2, x > 0, y \geq \frac{1}{x}\right\}$ and let d be the Euclidean metric on $\mathbb{R}^2$. Check whether X is complete or not as a metric subspace of $\mathbb{R}^2$.

   (b) Show that uniformly continuous image of a complete metric space is not necessarily complete. 3+2

5. Suppose d is a metric on a set X. Prove that the inequality
$$|d(x, y) - d(z, w)| \leq d(x, z) + d(y, w)$$
holds for all $x, y, z, w \in X$. Hence show that the function d is uniformly continuous on $X \times X$. 3+2

6. Let (X, d) be a metric space. Then prove that X is compact if and only if every collection of closed subsets of X having finite intersection property has non-empty intersection. 5

7. What do you mean by a 'contraction mapping' and a 'weak contraction mapping' of a metric space (X, d) into itself? Let $X = [1, \infty)$ and d be the usual metric on it. Define $T : X \to X$ by $T(x) = x + \frac{1}{x}$. Show that T is a weak contraction mapping on (X, d) but not a contraction mapping on (X, d). 2+3

8. (a) Suppose X is a connected metric space and $f : X \to \mathbb{R}$ is continuous. If $\alpha \in (\inf f(X), \sup f(X))$, then show that $\exists z \in X$ such that $f(z) = \alpha$.

(b) Let (X, d) be a connected metric space with X containing more than one point. Prove that X must be an infinite set. 2+3

9. (a) Find $d(A, B)$ for the sets $A = \left\{\left(x, x^2\right) : x \in [0, 1]\right\}$ and $B = \{(x, 1-x) : x \in [0, 1]\}$ with respect to the Euclidean metric on $\mathbb{R}^2$.

(b) Examine whether $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{\pi}{2} + x - \tan^{-1} x$ has a fixed point in $\mathbb{R}$. 3+2

Unit – 2

(Complex Analysis)

Answer ***any four*** questions.

10. Define stereographic projection. Show that the image of a line T under the stereographic projection is a circle minus north pole in the Riemann sphere S. 2+3

11. (a) Show that there does not exist any analytic function f such that $\mathrm{Im} f = x^3 - y^3$.

(b) Show that a Harmonic function $u(x, y)$ satisfies the differential equation $\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0$. 3+2

12. Show that the function $f(z) = e^{-z^{-4}} (z \neq 0), f(0) = 0$ is not analytic at $z = 0$, although Cauchy-Riemann equations are satisfied at the point $z = 0$. 5

13. Find a Mobius transformation, which maps the upper half plane $\{z : \mathrm{Im} z > 0\}$ onto itself, fixing only $0, \infty$. 5

14. Show that the sum function of a convergent power series in z is analytic in the interior of its circle of convergence. 5

15. (a) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius of convergence $R > 0$. Find the radius of convergence of the power series $\sum_{n=0}^{\infty} n^2 a_n z^n$.

(b) Evaluate the integral $\int_{|z-i|=1} \frac{z^2 dz}{z^2 + 1}$. 3+2

16. (a) Let C denote a contour of length L and suppose that a function $f(z)$ is piecewise continuous on C. If M is a non-negative constant such that $|f(z)| \leq M$ for all points z on C at which $f(z)$ is defined,

then prove that $\left|\int_C f(z)dz\right| \leq ML$.

(b) Without evaluating the integral show that

$$\left|\int_C \frac{dz}{z^4}\right| \leq 4\sqrt{2},$$

where C is the line segment from $z = i$ to $z = 1$. 3+2